Parking functions for mappings☆

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Abstract

We apply the concept of parking functions to functional digraphs of mappings by considering the nodes as parking spaces and the directed edges as one-way streets: Each driver has a preferred parking space and starting with this node he follows the edges in the graph until he either finds a free parking space or all reachable parking spaces are occupied. If all drivers are successful we speak of a parking function for the mapping. We transfer well-known characterizations of parking functions to mappings. Via analytic combinatorics techniques we study the total number $M_{n,m}$ of mapping parking functions, i.e., the number of pairs (f, s) with $f : [n] \to [n]$ an n-mapping and $s \in [n]^m$ a parking function for f with m drivers, yielding exact and asymptotic results. Moreover, we describe the phase change behaviour appearing at $m = n/2$ for $M_{n,m}$ and relate it to previously studied combinatorial contexts.

Keywords: Parking functions, mappings, enumeration, phase transition

1. Introduction

Parking functions were originally introduced by Konheim and Weiss [1] during their studies of the linear probing collision resolution scheme for hash tables. An illustrative definition can be given as follows: Consider a one-way street with n parking spaces numbered from 1 to n and a sequence of m drivers with preferred parking spaces $s_1, s_2, \ldots, s_m$. The drivers arrive sequentially and each driver k, $1 \leq k \leq m$, tries to park at his preferred parking space with address $s_k \in [n]$, where $[n] := \{1, 2, \ldots, n\}$. If it is free

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he parks. Otherwise he moves further in the allowed direction until he finds a free parking space. If there is no such parking space he leaves the street without parking. A sequence $(s_1, \ldots, s_m) \in [n]^m$ of addresses such that all drivers are able to park is then called a parking function. There are exactly $P_{n,m} = (n+1-m) \cdot (n+1)^{m-1}$ parking functions, for n parking spaces and $0 \leq m \leq n$ drivers [1].

Since their introduction, parking functions have been studied extensively and connections to various other combinatorial objects such as forests, hyperplane arrangements, acyclic functions and non-crossing partitions have been revealed [2]. Moreover, the notion of parking functions has been generalized in several ways, yielding, e.g., (a, b)-parking functions [3], bucket parking functions [4], x-parking functions [5], or G-parking functions [6].

Another natural generalization that has however not been studied yet is the following: Instead of considering simple one-way streets we allow road networks that are modelled by arbitrary directed graphs in which there is always exactly one possibility of moving forward, i.e., graphs in which every node has out-degree 1. Such graphs are the functional digraphs of mappings: Given a mapping $f : [n] \to [n]$ for some positive integer n, its functional digraph $G_f = (V, E)$ is defined on the vertices $V = [n]$ and has the edge set $E = \{(i, f(i)) : i \in [n]\}$. By considering the vertices as parking spaces and the edges one-way streets we obtain a natural generalization of parking functions to mappings. Again, every one of the $0 \leq m \leq n$ drivers has his preferred parking space $s_k \in [n]$ for $k \in [m]$ in the graph. The drivers arrive sequentially and each driver tries to park at his preferred parking space with address s_k. If it is empty he will park, otherwise he follows the edges and parks at the first empty node, if such one exists. Otherwise he cannot park since he would be caught in an endless loop. A pair (f, s) is then called an (n, m)-mapping parking function, if f is an n-mapping and $s \in [n]^m$ is a sequence of addresses such that all m drivers can park in the graph G_f. In Figure 1 we give an example of a mapping parking function.

To each (n, m)-mapping parking function (f, s) we associate its *output*-function $\pi = \pi_{(f,s)}$, with $\pi : [m] \to [n]$, where $\pi(k)$ is the address of the parking space in which the k-th driver ends up parking. Of course, π is an injection and for the particular case $m = n$ a bijection; thus in the latter case one may speak about the output-permutation π.

Obviously, the concept of mapping parking functions generalizes ordinary parking functions. Every ordinary parking function on $[n]$ can be identified with a parking function for the chain f that maps i to $(i + 1)$ for every $i \in [n - 1]$ and $f(n) = n$. Moreover, as a special case of mappings, we obtain parking functions on Cayley trees, i.e. rooted unordered labelled

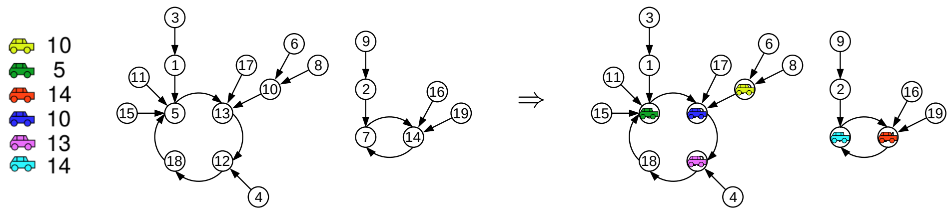

Figure 1: The functional digraph G_f of a 19-mapping f and a sequence $s = (10, 5, 14, 10, 13, 14)$ of addresses of preferred parking spaces for 6 drivers. All drivers are successful, thus (G_f, s) yields a $(19, 6)$-mapping parking function with output-function $\pi_{(f,s)}$ defined by the sequence $(10, 5, 14, 13, 12, 7)$ of parking positions of the drivers.

trees. To see this, let us recall the combinatorial structure of functional digraphs [7]: The weakly connected components are cycles of Cayley trees. That is, each connected component consists of Cayley trees (with edges oriented towards the root nodes) whose root nodes are connected by directed edges such that they form a cycle. We call a node j lying on a cycle, i.e., for which there exists a $k \geq 1$ such that $f^k(j) = j$, a cyclic node. Thus, a mapping consisting of a single connected component and having only one cyclic node corresponds to a Cayley tree (where a loop-edge has been added to the root).

We start our studies of parking functions for mappings and trees by exhibiting some of their basic properties and characterizations in Section 2.

The main focus of this paper lies on the exact and asymptotic enumeration of the total number of (n, m)-mapping parking functions. Let $\mathcal{M}_n := \{f : [n] \to [n]\}$ denote the combinatorial family of n-mappings for evry $n \in \mathbb{N}$. We are interested in the study of both the exact and asymptotic behaviour of the quantity

$$M_{n,m} := |\{(f, s) \,:\, f \in \mathcal{M}_n, s \in [n]^m \text{ and } s \text{ a parking function for } f\}|.$$

This treatment is divided into two main steps: First, in Section 3 we treat the important particular case $m = n$, i.e., we consider parking functions where the number of drivers is equal to the number of parking spaces. Second, the general case $0 \leq m \leq n$ is treated in Section 4. In order to get exact enumeration results we use suitable combinatorial decompositions of the objects, which give recursive descriptions of the quantities of interest. The recurrences occurring can be treated by a generating functions approach yielding partial differential equations. These differential equations allow for implicit characterizations of the generating functions via the solution of a certain functional equation (conceptually, such a treatment is related to [8]). Due to the combinatorial structure of functional digraphs, this treatment will first require the exact enumeration of parking functions for trees and we therefore introduce the following sequence:

$$F_{n,m} := |\{(T, s) \,:\, T \in \mathcal{T}_n, s \in [n]^m \text{ and } s \text{ a parking function for } T\}|,$$

where $\mathcal{T}_n \subseteq \mathcal{M}_n$ denotes the family of size-n Cayley trees.

From the exact results for $F_{n,m}$ and $M_{n,m}$ it follows somewhat surprisingly that $M_{n,m} = nF_{n,m}$, for $1 \le m \le n$, implying that, for fixed size n, the average number of parking functions per mapping is equal to the average number of parking function per tree. In Section 3 we construct a bijection providing a combinatorial explanation of this fact.

Asymptotic results for the case that the number of drivers coincides with the number of parking spaces can be obtained easily by applying standard singularity analysis of generating functions [7]. To give a complete picture of the asymptotic behaviour of $M_{n,m}$ depending on the growth of m w.r.t. n requires a more detailed study using saddle point methods. We consider the probability $p_{n,m} := M_{n,m}/n^{n+m}$ that a randomly chosen pair (f, s) of an n-mapping f and a sequence s of m addresses is indeed a parking function. For $m \sim \frac{n}{2}$ there occurs a phase change behaviour in this probability: If $\frac{m}{n} < \frac{1}{2} - \epsilon$, then there is asymptotically a positive probability that all drivers can park successfully, whereas for $\frac{m}{n} > \frac{1}{2} + \epsilon$ the probability that all drivers are successful is exponentially small. Qualitatively, the transient behaviour at $m \sim \frac{n}{2}$ is the same as observed previously in other combinatorial contexts, such as, e.g., in the analysis of random graphs during the phase where a giant component has not yet emerged. See [9, 10] or [7, Ch. VIII.10.].

In Section 5 we conclude this paper by giving some remarks on open problems and possible further research directions.

More detailed calculations as well as additional proof details can be found in the first author's PhD thesis [11].

2. Basic properties of parking functions for trees and mappings

Given an n-mapping f, we define a binary relation $\preceq_f$ on $[n]$ via

$$i \preceq_f j :\Longleftrightarrow \exists k \in \mathbb{N} : f^k(i) = j.$$

Thus $i \preceq_f j$ holds if there exists a directed path from i to j in the functional digraph G_f, and we say that j is a successor of i or that i is a predecessor of j. In this context a one-way street represents a total order.

Moreover, we will denote by $\mathsf{root}(T)$ the root of the Cayley tree T.

2.1. *Changing the order in a parking function*

In the setting of ordinary parking functions, changing the order in a sequence does not affect its property of being a parking function or not. This fact can easily be generalized to parking functions for mappings.

Lemma 2.1. *A function $s : [m] \to [n]$ is a parking function for a mapping $f : [n] \to [n]$ if and only if $s \circ \sigma$ is a parking function for f for any permutation σ on $[m]$.*

Proof. It is sufficient to prove the result for σ an elementary transposition $(k, k+1)$. Parking the $(k-1)$ first cars is the same for both s and $s \circ \sigma$. We consider the parking paths of the k-th and the $(k+1)$-th car in the mapping graph, i.e., the sequence of vertices between s_k (s_{k+1}) and $\pi_{(f,s)}(k)$ ($\pi_{(f,s)}(k+1)$). If these two paths are disjoint, the parking spaces of the k-th and $(k+1)$-th car are simply swapped in $s \circ \sigma$. If they are not disjoint, the k-th car will reach $\pi_{(f,s)}(k)$ before it reaches $\pi_{(f,s)}(k+1)$ and will thus park there. For the $(k+1)$-th car, the first free parking space after s_k is $\pi_{(f,s)}(k+1)$ and it will park there. In both cases, the k-th and the $(k+1)$-th car fill the same vertices $\pi_s(k)$ and $\pi_s(k+1)$ so the remaining cars parks at the same places as in s. □

2.2. *Alternative characterizations of parking functions*

Using the fact that one may reorder parking functions, one obtains the following well-known simpler characterization of ordinary parking functions $s : [n] \to [n]$ (see, e.g., [2]): A sequence $s \in [n]^n$ is a parking function if and only if it is a major function, i.e., the sorted rearrangement s' of the sequence s satisfies: $s'_j \leq j$ for all $j \in [n]$. In other words, there must be at least j elements in s that are not larger than j:

$$|\{k \in [n] : s_k \leq j\}| \geq j, \quad \text{for all } j \in [n]. \tag{1}$$

This characterization of parking functions can easily be generalized to parking functions for mappings. Indeed, in (1) we merely need to replace the $\leq$-relation on the integers $1, 2, \ldots, n$ by the binary relation given by the considered mapping:

Lemma 2.2. *Given an n-mapping f and a sequence $s \in [n]^n$, let $p(j)$ denote the number of predecessors of j, i.e., $p(j) := |\{i \in [n] : i \preceq_f j\}|$ and $q(j)$ denote the number of drivers whose preferred parking spaces are predecessors of j, i.e., $q(j) := |\{k \in [n] : s_k \preceq_f j\}|$. Then s is a mapping parking function for f if and only if*

$$q(j) \geq p(j), \quad \text{for all } j \in [n].$$

Proof. We denote by $P(j)$ the set of predecessors of j in the mapping f.

At most $q(j)$ cars may park in $P(j)$. Hence, if there exists a $j \in [n]$ such that $q(j) < p(j) = |P(j)|$, then at least one parking space remains free in $P(j)$ so s is not a parking function for f.

If at least $|P(j)| = p(j)$ cars have their preferred space in $P(j)$ then at least one car parks in vertex j: otherwise one would have $p(j)$ cars parked in $p(j) - 1$ vertices. Hence, $q(j) \geq p(j)$ means that a car is parked in vertex j. If this holds for all $j \in [n]$, a car is parked in every vertex of f and s is a parking function. □

Let us turn to parking functions, where the number of drivers does not necessarily coincide with the number of parking spaces. Equation (1) can be generalized as follows. A sequence $s : [m] \to [n]$ is an ordinary parking function if and only if

$$|\{k \in [m] : s_k \geq j\}| \leq n - j + 1, \quad \text{for all } j \in [n]. \tag{2}$$

We show how this can be generalized to mappings that are trees; the case of mappings in general is analogous but requires some more involved definitions.

Lemma 2.3. *Let T be a rooted labelled tree of size $|T| = n$ and s a sequence in $[n]^m$. Then s is a parking function for T if and only if*

$$\left|\left\{k \in [m] : s_k \in T'\right\}\right| \leq |T'|, \quad \textit{for all subtrees } T' \textit{ of } T \textit{ containing } \mathsf{root}(T).$$

Note that T' is called a subtree of T if T' is a subgraph of T that is a tree itself.

Proof. First, let T' be a subtree of T containing $\mathsf{root}(T)$. Then the possible parking spaces for a driver with $s_k \in T'$ all lie within T'. Thus, if the number $|\{k \in [m] : s_k \in T'\}|$ of such drivers exceeds the number of spaces $|T'|$ in T' at least one of the drivers will be unsuccessful and s is not a parking function for T.

Next, let us assume that s is not a parking function for T and that $\ell \in [m]$ is the first unsuccessful driver. Let T' be the maximal subtree of T containing $\mathsf{root}(T)$ and only such nodes that are occupied by one of the first $\ell - 1$ cars. Since T' is maximal all cars that are parked in T' wanted to park there and we have $|\{k \in [\ell - 1] : s_k \in T'\}| = |T'|$. Because the ℓ-th driver is unsuccessful, his preferred parking space is also in T', yielding $|\{k \in [\ell] : s_k \in T'\}| > |T'|$. □

2.3. Extremal cases for the number of parking functions

Given an n-mapping $f : [n] \to [n]$, let us denote by $S(f,m)$ the number of parking functions $s \in [n]^m$ for f with m drivers.

Proposition 2.4. *Let f and f' be two isomorphic n-mappings, i.e., there exists a bijective function $\sigma : [n] \to [n]$, such that $f' = \sigma \circ f \circ \sigma^{-1}$. Then for $0 \le m \le n$ it holds that*

$$S(f,m) = S(f',m).$$

Proof sketch. First note that the corresponding functional digraphs $G_f = ([n],E)$ and $G_{f'} = ([n],E')$ are isomorphic in the graph theoretic sense. It is then an easy task to show via induction that a function $s = (s_1,\ldots,s_m) \in [n]^m$ is a parking function for f if and only if $s' := \sigma \circ s = (\sigma(s_1),\ldots,\sigma(s_m))$ is a parking function for f'. □

In the following we consider the extremal cases of $S(f,m)$. Obviously, each injective function $s \in [n]^m$ is a parking function for every mapping $f \in \mathcal{M}_n$, which yields the trivial bounds

$$n^{\underline{m}} \le S(f,m) \le n^m, \quad \text{for } f \in \mathcal{M}_n.$$

These bounds are actually tight. Indeed, for the identity id_n on $[n]$, we have $S(\mathsf{id}_n,m) = n^{\underline{m}}$ since no collisions may occur. Moreover, for c a cycle of length n it holds that $S(c,m) = n^m$.

The situation becomes more interesting when we restrict ourselves to mappings that are trees. Let T be a rooted labelled tree and v a node of T. Furthermore, let U be a subtree of T attached to v such that $T \setminus U$ is still a tree. For a node w not contained in U, we denote by $\mathsf{reallocate}\Big(T \,\Big|\, \overset{U}{\underset{v}{\downarrow}} \mapsto \overset{U}{\underset{w}{\downarrow}}\Big)$ the tree operation of first detaching the subtree U from v and then attaching it to w. See Figure 2 for an illustration.

Lemma 2.5. *Let T be a rooted labelled tree and w a node in T. Furthermore, let v be a vertex on the path from w to $\mathsf{root}(T)$, U a subtree attached to v not containing w and $\tilde{T} = \mathsf{reallocate}\Big(T \,\Big|\, \overset{U}{\underset{v}{\downarrow}} \mapsto \overset{U}{\underset{w}{\downarrow}}\Big)$. Then it holds that*

$$S(\tilde{T},m) \ge S(T,m).$$

Proof. By applying Lemma 2.3 we will show that each parking function $s \in [n]^m$ for T is also a parking function for $\tilde{T}$. For this purpose, let s be a parking function for T and consider a subtree $\tilde{T}'$ of $\tilde{T}$ containing the root

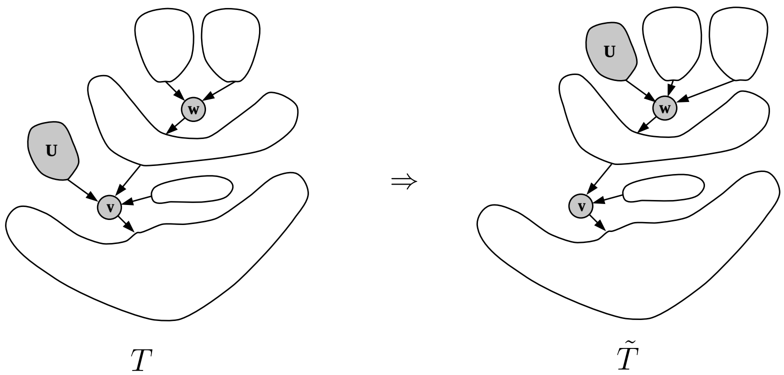

Figure 2: Illustrating the tree operation of reallocating the subtree U from v to w in T which yields the tree $\tilde{T}$.

of $\tilde{T}$. Note that by construction $\mathsf{root}(\tilde{T}) = \mathsf{root}(T)$. We distinguish between two cases to show that $\left|\left\{k \in [m] : s_k \in \tilde{T}'\right\}\right| \le |\tilde{T}'|$.

First, if $U \cap \tilde{T}' = \emptyset$ then $\tilde{T}'$ is also a subtree of T containing $\mathsf{root}(T)$. Since s is a parking function for T Lemma 2.3 implies that $\left|\left\{k \in [m] : s_k \in \tilde{T}'\right\}\right| \le |\tilde{T}'|$.

Second, if $U \cap \tilde{T}' = R \neq \emptyset$ then R is a subtree of U that is attached to node w, which is itself a predecessor of v. Within the tree $\tilde{T}'$, let us reallocate the subtree R from w to v. Then the resulting tree $T' := \mathsf{reallocate}\left(\tilde{T}' \,\middle|\, \underset{w}{\overset{R}{\downarrow}} \mapsto \underset{v}{\overset{R}{\downarrow}}\right)$ is a subtree of T containing $\mathsf{root}(T)$. According to Lemma 2.3 it holds that $|\{k \in [m] : s_k \in T'\}| \le |T'|$. Since T' and $\tilde{T}'$ have equal size and the nodes in the corresponding trees have the same labels, this also implies that $\left|\left\{k \in [m] : s_k \in \tilde{T}'\right\}\right| \le |\tilde{T}'|$.

□

With this lemma we can easily obtain tight bounds on $S(T, m)$.

Theorem 2.6. *Let* star_n *be the rooted labelled tree of size* n *with root node* n *and all other nodes attached to it. Furthermore let* chain_n *be the tree with root node* n *and node* j *attached to node* $(j+1)$, *for* $1 \le j \le n-1$. *Then, for any rooted labelled tree* T *of size* n *it holds*

$$S(\mathsf{star}_n, m) \le S(T, m) \le S(\mathsf{chain}_n, m), \tag{3}$$

yielding the bounds

$$n^{\underline{m}} + \binom{m}{2}(n-1)^{\underline{m-1}} \leq S(T,m) \leq (n-m+1)(n+1)^{m-1}, \quad \textit{for } 0 \leq m \leq n.$$

Proof. Each tree T of size n can be constructed from a tree T_0, which is isomorphic to star_n, by applying a sequence of reallocations $T_{i+1} := \mathsf{reallocate}\left(T_i \,\middle|\, \substack{U_i \\ \downarrow \\ v_i} \mapsto \substack{U_i \\ \downarrow \\ w_i}\right)$, with $w_i \preceq_{T_i} v_i$, for $0 \leq i \leq k$, with $k \geq 0$. Furthermore, starting with $T =: \tilde{T}_0$, there always exists a sequence of reallocations $\tilde{T}_{i+1} := \mathsf{reallocate}\left(\tilde{T}_i \,\middle|\, \substack{\tilde{U}_i \\ \downarrow \\ \tilde{v}_i} \mapsto \substack{\tilde{U}_i \\ \downarrow \\ \tilde{w}_i}\right)$, with $\tilde{w}_i \preceq_{\tilde{T}_i} \tilde{v}_i$, for $0 \leq i \leq \tilde{k}$, with $\tilde{k} \geq 0$, such that the resulting tree is isomorphic to chain_n. Thus, equation (3) follows immediately from Lemma 2.5. The upper bound is the well-known formula for the number ordinary parking functions. Elementary combinatorics yield the number of parking functions with m drivers for star_n. $\square$

3. Total number of parking functions: number of drivers coincides with number of parking spaces

Due to the combinatorial structure of mapping graphs, we start this section with the study of parking functions for trees. Then we proceed to connected mappings and finally to the general case.

3.1. Tree parking functions

We study the total number $F_n := F_{n,n}$ of (n,n)-tree parking functions, i.e., the number of pairs (T,s), with $T \in \mathcal{T}_n$ a Cayley tree of size n and $s \in [n]^n$ a parking sequence of length n for the tree T, such that all drivers are successful. To obtain a recursive description of the total number F_n of tree parking functions we use the decomposition of a Cayley tree $T \in \mathcal{T}_n$ w.r.t. the last empty node. We thus consider the situation just before the last driver starts searching a parking space.

Two different situations might occur: (i) the empty node is the root node of the tree T, or (ii) the empty node is a non-root node. See Figure 3 for a schematic representation. In case (i) the last driver will find a free parking space regardless of the n possible choices of his preferred parking space. In case (ii) the last driver will only find a free parking space, if his preferred parking space lies in the subtree T'' rooted at the node corresponding to the free parking space. If we detach the edge linking T'' with the rest of the tree we get a pair of unordered trees; let us assume T', the tree containing the original root of the tree, has size k whereas T'' has size $n-k$. Then there are

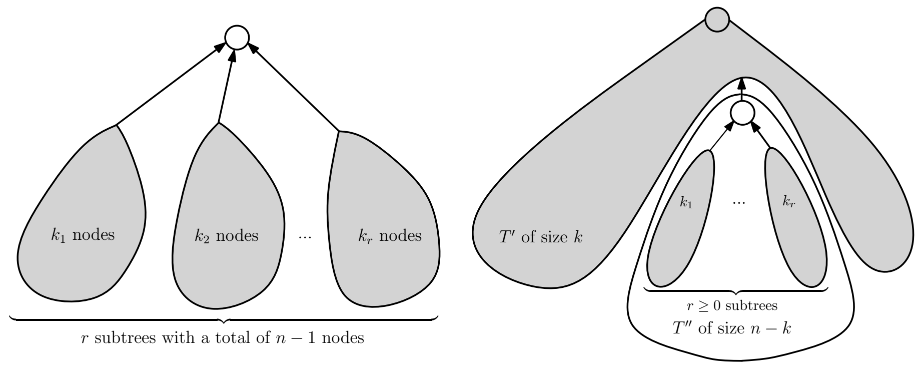

Figure 3: Schematic representation of the two situations that might occur when considering parking functions with n drivers for a Cayley tree with n nodes. The last empty node is marked in white.

$n-k$ choices for the preferred parking space of the last driver. Furthermore, it is important to take into account that, given T' and T'', the original tree T cannot be reconstructed, since there are always k different trees in $\mathcal{T}_n$ leading to the same pair (T', T''). Considering the order-preserving relabellings of the subtrees and the merging of the parking sequences for the subtrees, we obtain the following recursive description of F_n.

$$F_n = \sum_{r \geq 1} \frac{1}{r!} \sum_{\substack{\sum_{i=1}^{r} k_i = n-1 \\ k_i \geq 1}} F_{k_1} \cdot F_{k_2} \cdot \dots \cdot F_{k_r} \binom{n}{k_1, k_2, \dots, k_r, 1} \binom{n-1}{k_1, k_2, \dots, k_r} n$$
$$+ \sum_{r \geq 0} \frac{1}{r!} \sum_{\substack{k + \sum_{i=1}^{r} k_i = n-1 \\ k \geq 1, k_i \geq 1}} F_k \cdot F_{k_1} \cdot F_{k_2} \cdot \dots \cdot F_{k_r} \cdot$$
$$\cdot \binom{n}{k, k_1, k_2, \dots, k_r, 1} \binom{n-1}{k, k_1, k_2, \dots, k_r} k(n-k),$$

for $n \geq 2$ with initial value $F_1 = 1$. Here r denotes the number of subtrees of the empty node and the factor $\frac{1}{r!}$ occurs since every one of the $r!$ orderings of the subtrees represents the same tree. Introducing the generating function $F(z) := \sum_{n \geq 1} F_n \frac{z^n}{(n!)^2}$ the recurrence relation for F_n can be transferred into the following differential equation:

$$F'(z) = \exp(F(z)) \cdot \left(1 + zF'(z)\right)^2, \quad F(0) = 0. \tag{4}$$

Solving this differential equation can be done with standard methods and it can be checked easily that the solution of (4) is given as follows:

$$F(z) = T(2z) + \ln\left(1 - \frac{T(2z)}{2}\right), \tag{5}$$

where $T(z) := \sum_{n\geq 1} T_n \frac{z^n}{n!} = \sum_{n\geq 1} n^{n-1}\frac{z^n}{n!}$ denotes the tree function, i.e., the exponential generating function of the number $T_n = n^{n-1}$ of Cayley trees of size n . Recall that it satisfies the functional equation

$$T(z) = ze^{T(z)} \tag{6}$$

and is thus related to the so-called Lambert W-function [7].

3.2. Mapping parking functions

Now we turn to the total number $M_n := M_{n,n}$ of (n, n)-mapping parking functions. We first introduce connected mappings as auxiliary objects and study parking functions for them; after that the general situation can be treated easily. An n-mapping f is simply a set of connected mappings whose respective sizes add up to n. This relation between mappings and connected mappings can be translated immediately into connections between parking functions for these objects. However, this is not the case for connected mappings and trees. Indeed, the decomposition of connected mappings $\mathcal{C}$ into Cayley trees $\mathcal{T}$ is not consistent with the parking procedure. Instead of using this decomposition, we will therefore apply a decomposition of connected mappings w.r.t. the last empty node in the parking procedure. So, let us introduce the total number C_n of parking functions of length n for connected n-mappings.

Three situations may occur: (i) the last empty node is the root node of the Cayley tree that forms a length-1 cycle, (ii) the last empty node is the root node of a Cayley tree lying in a cycle of at least two trees, (iii) the last empty node is not a cyclic node, i.e., it is not one of the root nodes of the Cayley trees forming the cycle. A schematic representation of these situations can be found in Figure 4.

To treat these cases only slight adaptions to the considerations made in Section 3.1 have to be done; case (i) is explained already there. In case (ii) the last driver will find a free parking space regardless of the n possible choices of his preferred parking space. Let us denote by T'' the tree whose root node is the last free parking space. When we detach the two edges linking T'' with the rest of the mapping graph, we cut the cycle and the graph decomposes into a pair of trees: the tree T'' and the unordered tree T', which we may consider rooted at the former predecessor of the free

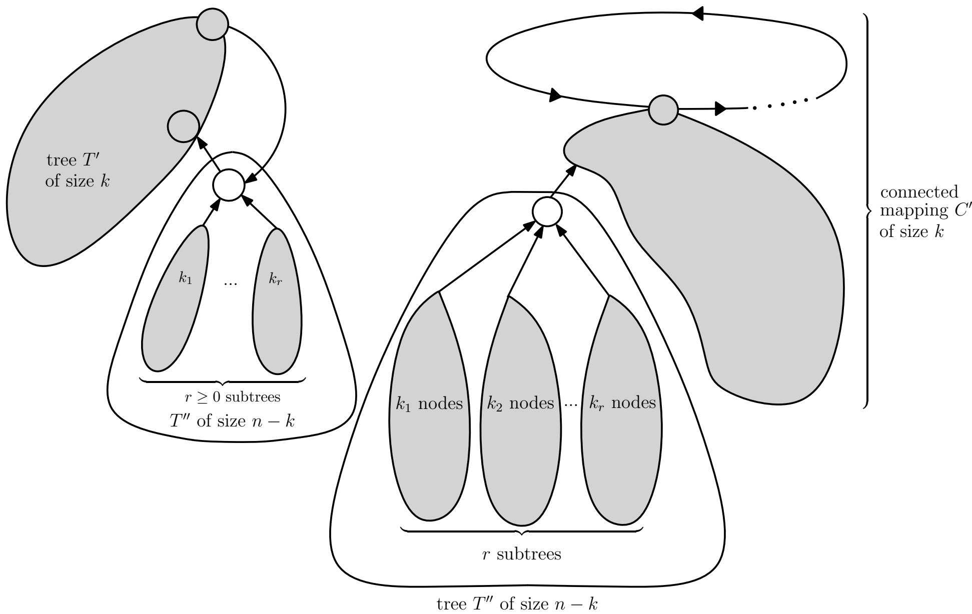

Figure 4: Schematic representation of two of the three situations that might occur when considering parking functions with n drivers for a connected n-mapping. The last empty node is marked in white; the case where it forms the root node of the single tree constituting the mapping is depicted on the left-hand side of Figure 3.

parking space in the cycle of the original graph. In case (iii) the last driver will only find a free parking space if his preferred parking space is contained in T''. If we detach the edge linking this subtree T'' with the rest of the graph, a connected mapping graph remains (call it C'). Taking into account the order-preserving relabellings of the substructures and also the merging of the parking sequences for them, we obtain the following recursive description of C_n, valid for all $n \geq 1$:

$$\begin{aligned} C_n = &\sum_{r\geq 0} \frac{1}{r!} \sum_{\sum k_i = n-1} F_{k_1} F_{k_2} \cdots F_{k_r} \binom{n}{k_1, k_2, \ldots, k_r} \binom{n-1}{k_1, k_2, \ldots, k_r} \cdot n \\ &+ \sum_{r\geq 0} \frac{1}{r!} \sum_{k+\sum k_i = n-1} F_k F_{k_1} \cdots F_{k_r} \binom{n}{k, k_1, \ldots, k_r} \binom{n-1}{k, k_1, \ldots, k_r} \cdot kn \\ &+ \sum_{r\geq 0} \frac{1}{r!} \sum_{k+\sum k_i = n-1} C_k F_{k_1} \cdots F_{k_r} \binom{n}{k, k_1, \ldots, k_r} \binom{n-1}{k, k_1, \ldots, k_r} \cdot k(n-k). \end{aligned} \tag{7}$$

Introducing the generating function $C(z) := \sum_{n\geq 1} C_n \frac{z^n}{(n!)^2}$, the recurrence (7) yields the following differential equation for $C(z)$,

$$C'(z) \cdot \big(1 - z\exp(F(z)) - z^2 F'(z)\exp(F(z))\big) \\ = \Big(\big(1 + zF'(z)\big)^2 + zF'(z) + z^2 F''(z)\Big)\exp(F(z)), \quad C(0) = 0, \quad (8)$$

where $F(z)$ denotes the generating function of the number of tree parking functions given in (5). This differential equation has the following simple solution:

$$C(z) = \ln\left(\frac{1}{1 - \frac{T(2z)}{2}}\right), \qquad (9)$$

as can be checked easily by using the functional equation (6) of the tree function $T(z)$.

Now we are in the position to study the total number M_n of (n,n)-mapping parking functions. Again we introduce the generating function $M(z) := \sum_{n\ge 0} M_n \frac{z^n}{(n!)^2}$. Since the functional digraph of a mapping can be considered as the set of its connected components and furthermore a parking function for a mapping can be considered as a shuffle of the corresponding parking functions for the connected components, we get the following simple relation between the generating functions $M(z)$ and $C(z)$: $M(z) = \exp(C(z))$. Thus, by using (9), the generating function $M(z)$ is given as follows:

$$M(z) = \frac{1}{1 - \frac{T(2z)}{2}}. \qquad (10)$$

Next, we remark that the following relation between $M(z)$ and $F(z)$, the generating functions for the number of parking functions for mappings and trees, holds:

$$1 + zF'(z) = 1 + \frac{T(2z)}{1 - T(2z)} \cdot \left(1 - \frac{1}{2 - T(2z)}\right) = 1 + \frac{\frac{T(2z)}{2}}{1 - \frac{T(2z)}{2}} = M(z).$$

At the level of coefficients, this immediately shows the following somewhat surprising connection between F_n and M_n.

Theorem 3.1. *For all $n \ge 1$ it holds that the total numbers F_n and M_n of (n,n)-tree parking functions and (n,n)-mapping parking functions, respectively, satisfy:*

$$M_n = n \cdot F_n.$$

Since the number of mappings of size n is exactly n times the number of Cayley trees of size n, this implies that the average number of parking functions per mapping of a given size is exactly equal to the average number of parking functions per tree of the same size. Later, in Section 3.3 we establish a combinatorial explanation for this interesting fact.

Extracting coefficients from the generating function solution (10) of $M(z)$ easily yields exact formulæ for M_n.

Theorem 3.2. *The total number M_n of (n, n)-mapping parking functions is for $n \geq 1$ given as follows:*

$$M_n = n!(n-1)! \cdot \sum_{j=0}^{n-1} \frac{(n-j) \cdot (2n)^j}{j!}.$$

Proof. Using (6) and the Lagrange inversion formula [2], we obtain

$$\begin{aligned}
[z^n] \frac{1}{1 - \frac{T(2z)}{2}} &= \frac{2^n}{n} [T^{n-1}] \frac{e^{nT}}{2\left(1 - \frac{T}{2}\right)^2} \\
&= \frac{2^{n-1}}{n} \sum_{k=0}^{n-1} \frac{(k+1) n^{n-1-k}}{2^k (n-1-k)!} = \frac{1}{n} \sum_{j=0}^{n-1} \frac{(n-j)(2n)^j}{j!},
\end{aligned}$$

and thus

$$M_n = (n!)^2 [z^n] M(z) = n!(n-1)! \sum_{j=0}^{n-1} \frac{(n-j) \cdot (2n)^j}{j!}.$$

□

The asymptotic behaviour of the numbers M_n can be deduced from the generating function solution (10) of $M(z)$. Using the well-known asymptotic expansion of the tree function $T(z)$ in a complex neighbourhood of its unique dominant singularity $\frac{1}{e}$ (see [7]),

$$T(z) = 1 - \sqrt{2}\sqrt{1 - ez} + \frac{2}{3}(1 - ez) + \mathcal{O}((1 - ez)^{\frac{3}{2}}), \tag{11}$$

one immediately obtains that $M(z)$ inherits a singularity from $T(z)$ at $\rho = \frac{1}{2e}$. This is the unique singularity of $M(z)$, since $T(2z) = 2$ has no solution. Its local expansion in a complex neighbourhood of ρ can easily be obtained as follows:

$$M(z) = \frac{2}{2 - T(2z)} = \frac{2}{1 + \sqrt{2}\sqrt{1 - 2ez} - \frac{2}{3}(1 - 2ez) + \mathcal{O}((1 - 2ez)^{\frac{3}{2}})}$$

$$= 2 - 2\sqrt{2}\sqrt{1-2ez} + \frac{16}{3}(1-2ez) + \mathcal{O}((1-2ez)^{\frac{3}{2}}).$$

A standard application of singularity analysis of generating functions, i.e., transfer lemmata [7] which allow to deduce the asymptotic behaviour of the coefficients from the local behaviour of the generating function around its dominant singularity, shows the following asymptotic equivalent of the numbers M_n. We get

$$[z^n]M(z) \sim \frac{\sqrt{2}}{\sqrt{\pi}} \frac{(2e)^n}{n^{\frac{3}{2}}}$$

and the following corollary, which follows directly when applying Stirling's approximation formula for the factorials [7].

Corollary 3.3. *The total number M_n of (n,n)-mapping parking functions is asymptotically, for $n \to \infty$, given as follows:*

$$M_n \sim \frac{\sqrt{2\pi}\, 2^{n+1} n^{2n}}{\sqrt{n}\, e^n}.$$

3.3. Bijective relation between parking functions for trees and mappings

The simple relation between the total number of parking functions for trees and mappings stated in Theorem 3.1 was proved by algebraic manipulations of the corresponding generating functions. This does not provide a combinatorial explanation of this fact. However, standard constructions such as Prüfer codes do not seem to be applicable to this setting. We thus present a bijective proof of this result in the following.

Theorem 3.4. *For each $n \geq 1$, there exists a bijection φ from the set of triples (T, s, w), with $T \in \mathcal{T}_n$ a tree of size n, $s \in [n]^n$ a parking function for T with n drivers, and $w \in T$ a node of T, to the set of pairs (f, s) where $f \in \mathcal{M}_n$ is an n-mapping and $s \in [n]^n$ is a parking function for f with n drivers. Thus*

$$n \cdot F_n = M_n, \quad \text{for } n \geq 1.$$

Note that indeed s remains fixed under the bijection φ. The map φ is illustrated in Figure 5 where an example involving a tree of size 8 is given.

Proof. First, we define the rank $k(v)$ of a node v to be $\pi^{-1}(v)$, where the output-function π of (T, s) is a bijection since s is a parking function for T with n drivers. That is, $k(v) = i$ if and only if the i-th car in the parking sequence ends up parking at node v in T. For an example, see the second picture in Figure 5. Furthermore, we will denote by $T(v)$ the parent of node

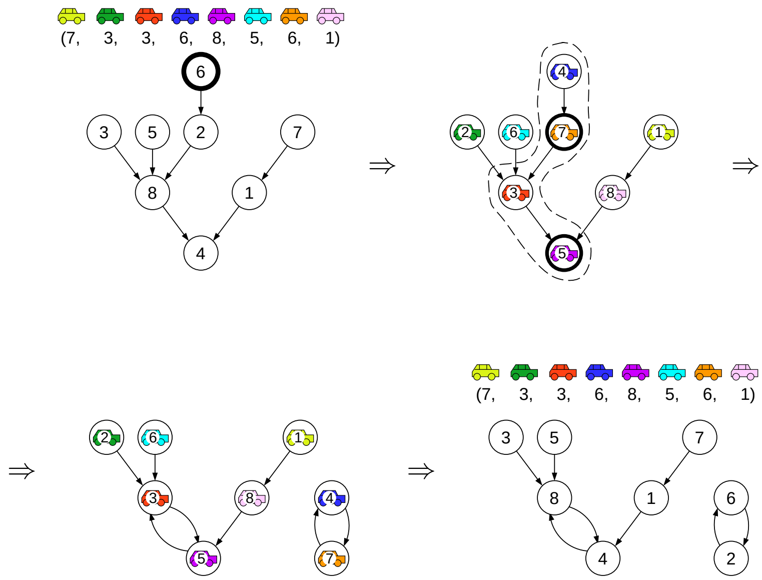

Figure 5: The bijection φ described in Theorem 3.4 is applied to the triple (T, s, w) represented in the top left corner and yields the mapping parking function represented in the bottom right corner. The labels of the cars denote their ranks; the marked nodes in the second picture correspond to the right-to-left maxima in the sequence of ranks of the drivers on the path from w to the root.

v in the tree T. That is, for $v \neq \mathsf{root}(T)$, $T(v)$ is the unique node such that $(v, T(v))$ is an edge in T.

Given a triple (T, s, w), we consider the unique path $w \rightsquigarrow \mathsf{root}(T)$ from the node w to the root of T. It consists of the nodes $v_1 = w$, $v_2 = T(v_1), \dots, v_{i+1} = T(v_i), \dots, v_r = \mathsf{root}(T)$ for some $r \geq 1$. To this sequence $v_1, v_2, \dots, v_r$ of nodes in T we associate its sequence of ranks $k_1, \dots, k_r$ where $k_i := k(v_i)$. We denote by $I = (i_1, \dots, i_t)$, with $i_1 < i_2 < \cdots < i_t$ for some $t \geq 1$, the indices of the right-to-left maxima in this sequence, i.e.,

$$i \in I \iff k_i > k_j, \quad \text{for all } j > i.$$

The corresponding set of nodes in the path $w \rightsquigarrow \mathsf{root}(T)$ will be denoted by $V_I := \{v_i : i \in I\}$. Of course, if follows from the definition that the root node is always contained in V_I, i.e., $v_r \in V_I$. Note that the idea of this proof is inspired by the construction of the *fundamental bijection* [2] showing that right-to-left maxima and cycles in permutations are equi-distributed.

We can now describe the function φ by constructing an n-mapping f, such that s is a parking function for f. In case $v \notin V_I$ we simply set $f(v) = T(v)$. If $v \in V_I$ we have $v = v_{i_\ell}$ for some $1 \leq \ell \leq t$. The crucial observation is that the edge $(v_{i_\ell}, T(v_{i_\ell}))$, is never used by any of the drivers of s. Since k_{i_ℓ} is a right-to-left maximum in the sequence $k_1, \dots, k_r$, all nodes that lie on the path from v_{i_ℓ} to the root are already occupied when the k_{i_ℓ}-th driver parks at v_{i_ℓ}. Thus, no driver before k_{i_ℓ} (then he would have parked at v_{i_ℓ}) nor after k_{i_ℓ} (then he would not be able to park anywhere) could have reached and thus left the node v_{i_ℓ}. We may thus delete this edge and attach the node v_{i_ℓ} to an arbitrary node without violating the property that s is a parking function. Since we want to be able to reconstruct T from f we will do this in the following way: $f(v_{i_\ell}) := T\left(v_{i_{\ell-1}}\right)$, where we set $T(v_{i_0}) = v_1 = w$. This means that the nodes on the path $w \rightsquigarrow \mathsf{root}(T)$ in T form t cycles $C_1 := (v_1, \dots, v_{i_1}), \dots, C_t := (T(v_{i_{t-1}}), \dots, v_r = v_{i_t})$ in G_f.

Having defined the mapping f in this way, the sequence s is also a parking function for f and it holds that the parking paths of the drivers coincide for T and f. In particular, it holds that $\pi_{(f,s)} = \pi_{(T,s)}$.

Moreover, it is easy to describe the inverse function φ^{-1}. Given a pair (f, s), we start by computing the rank of every node in G_f. Then we sort the connected components of G_f in decreasing order of their cyclic elements with highest rank. That is, if G_f consists of t connected components and c_i denotes the cyclic element in the i-th component with highest rank, we have $k(c_1) > k(c_2) > \dots > k(c_t)$. Then, for every $1 \leq i \leq t$, we remove the edges (c_i, d_i) where $d_i = f(c_i)$. Next we reattach the components to each other by establishing the edges (c_i, d_{i+1}) for every $1 \leq i \leq t-1$. This leads to the tree T. Note that the node c_t is attached nowhere since it constitutes the root of T. Setting $w = d_1$, we obtain the preimage (T, s, w) of (f, s). □

4. Total number of parking functions: the general case

In this section we study the exact and asymptotic behaviour of the total number of mapping parking functions for the general case of n parking spaces and $0 \leq m \leq n$ drivers. In what follows we use $\tilde{m} := n - m$, i.e., $\tilde{m}$ denotes the number of empty parking spaces in the mapping graph after all m drivers have parked. Again, we start with the number of parking functions for trees.

4.1. Tree parking functions

In the following, we use the notation $\tilde{F}_{n,\tilde{m}} := F_{n,n-\tilde{m}}$, thus $\tilde{F}_{n,0} = F_n$. Let us assume that $1 \leq \tilde{m} \leq n$. To get a recursive description for the numbers $\tilde{F}_{n,\tilde{m}}$, we use the combinatorial decomposition of a Cayley tree

$T \in \mathcal{T}_n$ w.r.t. the free node which has the largest label amongst all $\tilde{m}$ empty nodes in the tree.

Again, the two situations depicted in Figure 3 have to be considered. The argument given in Section 3.1 for the case $\tilde{m} = 0$ can be adapted easily: in case (i), the root node is the empty node with largest label and we assume that the r subtrees of the root are of sizes $k_1, \dots, k_r$ and contain $\ell_1, \dots, \ell_r$ empty nodes, respectively. In case (ii), a non-root node is the empty node with largest label. We denote by T'' the subtree of T rooted at this empty node. After detaching T'' from the remaining tree we obtain a tree T' that is of size k and has ℓ empty nodes for some $1 \le k \le n-1$ and $0 \le \ell \le \tilde{m}-1$. Furthermore, we assume that the r subtrees of the root of T'' are of sizes $k_1, \dots, k_r$ and contain $\ell_1, \dots, \ell_r$ empty nodes, respectively. In the latter case one has to take into account that there are k possibilities of attaching the root of T'' to one of the k nodes in T' yielding the same decomposition. The following recursive description of the numbers $\tilde{F}_{n,\tilde{m}}$ follows by considering the order-preserving relabellings of the subtrees and also the merging of the parking sequences for the subtrees. Moreover, one uses the simple fact that, when fixing an empty node v and considering all possible labellings of the $\tilde{m}$ empty nodes, only a fraction of $\frac{1}{\tilde{m}}$ of all labellings leads to v having the largest label amongst all empty nodes.

We then get the following recurrence

$$\begin{aligned}
\tilde{F}_{n,\tilde{m}} = \frac{1}{\tilde{m}} \sum_{r\ge 0} \frac{1}{r!} \sum_{k_1+\dots+k_r=n-1} \sum_{\ell_1+\dots+\ell_r=\tilde{m}-1} \tilde{F}_{k_1,\ell_1} \cdot \tilde{F}_{k_2,\ell_2} \cdots \tilde{F}_{k_r,\ell_r} \cdot \\
\cdot \binom{n}{k_1, k_2, \dots, k_r} \binom{n-\tilde{m}}{k_1-\ell_1, k_2-\ell_2, \dots, k_r-\ell_r} \\
+ \frac{1}{\tilde{m}} \sum_{r\ge 0} \frac{1}{r!} \sum_{k+k_1+\dots+k_r=n-1} \cdot \sum_{\ell+\ell_1+\dots+\ell_r=\tilde{m}-1} \tilde{F}_{k,\ell} \tilde{F}_{k_1,\ell_1} \cdots \tilde{F}_{k_r,\ell_r} \cdot \\
\cdot \binom{n}{k, k_1, \dots, k_r} \binom{n-\tilde{m}}{k-\ell, k_1-\ell_1, \dots, k_r-\ell_r} \cdot k, \quad \text{for } 1 \le \tilde{m} \le n,
\end{aligned} \tag{12}$$

with initial values $\tilde{F}_{n,0} = F_n$. We introduce the generating function

$$\tilde{F}(z,u) := \sum_{n\ge 1} \sum_{\tilde{m}\ge 0} \tilde{F}_{n,\tilde{m}} \frac{z^n u^{\tilde{m}}}{n!(n-\tilde{m})!} = \sum_{n\ge 1} \sum_{0\le m\le n} F_{n,n-m} \frac{z^n u^{n-m}}{n! m!}. \tag{13}$$

The recurrence relation (12) then yields, after straightforward computations, the following partial differential equation for $\tilde{F}(z,u)$:

$$\tilde{F}_u(z,u) = z^2 \tilde{F}_z(z,u) \exp(\tilde{F}(z,u)) + z \exp(\tilde{F}(z,u)), \tag{14}$$

with initial condition $\tilde{F}(z,0) = F(z)$ and $F(z) = \sum_{n\geq 1} F_n \frac{z^n}{(n!)^2}$ given by (5). A suitable representation of the solution of this PDE as given next is crucial for further studies.

Proposition 4.1. *The generating function $\tilde{F}(z,u)$ defined in (13) is given by*

$$\tilde{F}(z,u) = Q \cdot (2 + u(1-Q)) + \ln(1-Q) = \ln\left(\frac{Q(1-Q)}{z}\right),$$

where the function $Q = Q(z,u)$ is given implicitly as the solution of the functional equation

$$Q = z \cdot e^{Q\cdot(2+u(1-Q))}. \tag{15}$$

The solution of the PDE in (14) can be found using the "method of characteristics" [12]; these steps are detailed in the first author's thesis [11]. Checking that the above function is indeed a solution can be done easily.

4.2. Mapping parking functions

As pointed out in Section 3.2, it suffices to provide the relevant considerations for the subfamily $\mathcal{C}_n$ of connected n-mappings, since results for the general situation can then be deduced easily. Thus, let us introduce the total number $C_{n,m}$ of parking functions of length m for connected n-mappings.

Let us consider parking functions for connected mappings for the case that $\tilde{m} = n - m$ parking spaces remain free after all drivers have parked successfully. We define $\tilde{C}_{n,\tilde{m}} := C_{n,n-\tilde{m}}$. In order to obtain a recursive description of the numbers $\tilde{C}_{n,\tilde{m}}$ we again use the combinatorial decomposition of a connected mapping $f \in \mathcal{C}_n$ w.r.t. the free node which has the largest label amongst all $\tilde{m}$ empty nodes in the mapping graph.

As for the case $\tilde{m} = 0$, three situations may occur when using this decomposition: (i) the empty node with largest label is the root node of the Cayley tree which forms a length-1 cycle, (ii) the empty node with largest label is the root node of a Cayley tree forming a cycle of at least two trees and (iii) the empty node with largest label is not a cyclic node. Analogous considerations to the ones given for tree parking functions in Section 4.1 lead to the following recursive description of the number of parking functions for connected mappings for $1 \leq \tilde{m} \leq n$:

$$\tilde{C}_{n,\tilde{m}} = \frac{1}{\tilde{m}} \sum_{r\geq 0} \frac{1}{r!} \sum_{k_1+\cdots+k_r=n-1} \sum_{\ell_1+\cdots+\ell_r=\tilde{m}-1} \tilde{F}_{k_1,\ell_1} \cdot \tilde{F}_{k_2,\ell_2} \cdots \tilde{F}_{k_r,\ell_r} \cdot \\ \cdot \binom{n}{k_1, k_2, \ldots, k_r} \binom{n-\tilde{m}}{k_1-\ell_1, k_2-\ell_2, \ldots, k_r-\ell_r}$$

$$+\frac{1}{\tilde{m}}\sum_{r\geq 0}\frac{1}{r!}\sum_{k+k_1+\cdots+k_r=n-1}\ \sum_{\ell+\ell_1+\cdots+\ell_r=\tilde{m}-1}\tilde{F}_{k,\ell}\tilde{F}_{k_1,\ell_1}\cdots\tilde{F}_{k_r,\ell_r}\cdot \quad (16)$$
$$\cdot\binom{n}{k,k_1,\ldots,k_r}\binom{n-\tilde{m}}{k-\ell,k_1-\ell_1,\ldots,k_r-\ell_r}\cdot k$$
$$+\frac{1}{\tilde{m}}\sum_{r\geq 0}\frac{1}{r!}\sum_{k+k_1+\cdots+k_r=n-1}\ \sum_{\ell+\ell_1+\cdots+\ell_r=\tilde{m}-1}\tilde{C}_{k,\ell}\tilde{F}_{k_1,\ell_1}\cdots\tilde{F}_{k_r,\ell_r}\cdot$$
$$\cdot\binom{n}{k,k_1,\ldots,k_r}\binom{n-\tilde{m}}{k-\ell,k_1-\ell_1,\ldots,k_r-\ell_r}\cdot k,$$

with initial values $\tilde{C}_{n,0}=C_n$. When introducing the generating function

$$\tilde{C}(z,u):=\sum_{n\geq 1}\sum_{\tilde{m}\geq 0}\tilde{C}_{n,\tilde{m}}\frac{z^n u^{\tilde{m}}}{n!(n-\tilde{m})!}, \quad (17)$$

recurrence (16) yields the following first order linear partial differential equation for the function $\tilde{C}(z,u)$:

$$\tilde{C}_u(z,u)=z^2\tilde{C}_z(z,u)\exp(\tilde{F})+z\exp(\tilde{F})+z^2\tilde{F}_z\exp(\tilde{F}), \quad (18)$$

with $\tilde{F}=\tilde{F}(z,u)$ the corresponding generating function for the number of tree parking functions given in Proposition 4.1, and initial condition $\tilde{C}(z,0)=C(z)$, with $C(z)=\sum_{n\geq 1}C_n\frac{z^n}{(n!)^2}$ given by (9).

Proposition 4.2. *The generating function $\tilde{C}(z,u)$ defined in (17) is given as follows:*

$$\tilde{C}(z,u)=\ln\left(\frac{1}{(1-Q)(1-uQ)}\right),$$

where the function $Q=Q(z,u)$ is given implicitly as the solution of the functional equation (15).

Proof sketch. First, one shows that the function $Q(z,u)$ defined by equation (15) fulfils $Q_u(z,u)=z^2Q_z(z,u)e^{\tilde{F}(z,u)}$, i.e., $Q(z,u)$ solves the reduced PDE corresponding to (18). This suggests the substitution $z=z(Q):=Q/e^{Q(2+u(1-Q))}$ and we introduce

$$\hat{C}(Q,u):=\tilde{C}(z(Q),u)=\tilde{C}\left(\frac{Q}{e^{Q(2+u(1-Q))}},u\right).$$

Equation (18) then reads as $\hat{C}_u(Q,u)=\frac{Q}{1-uQ}$. After back-substitution, the general solution of this equation is given by

$$\tilde{C}(z,u)=\ln\left(\frac{1}{1-uQ(z,u)}\right)+\tilde{h}(Q(z,u)),$$

with an arbitrary differentiable function $\tilde{h}(x)$. This function can be characterized by evaluating $\tilde{C}(z,u)$ at $u = 0$ and using the initial condition $\tilde{C}(z,u) = C(z)$. □

We are now able to treat the total number $M_{n,m}$ of (n,m)-mapping parking functions. We introduce $\tilde{M}_{n,\tilde{m}} := M_{n,n-\tilde{m}}$ and the generating function

$$\tilde{M}(z,u) := \sum_{n\geq 0}\sum_{\tilde{m}\geq 0} \tilde{M}_{n,\tilde{m}} \frac{z^n u^{\tilde{m}}}{n!(n-\tilde{m})!}. \tag{19}$$

The decomposition of mapping parking functions into parking functions for their connected components immediately gives the relation $\tilde{M}(z,u) = \exp\left(\tilde{C}(z,u)\right)$. According to Proposition 4.2 we obtain the following solution of $\tilde{M}(z,u)$.

Proposition 4.3. *The generating function $\tilde{M}(z,u)$ defined in (19) is given as follows:*

$$\tilde{M}(z,u) = \frac{1}{(1-Q)(1-uQ)},$$

where the function $Q = Q(z,u)$ is defined implicitly by the functional equation (15).

Using the representations of the generating functions $\tilde{F}(z,u)$ and $\tilde{M}(z,u)$ for the number of tree and mapping parking functions given in Proposition 4.1 and 4.3, respectively, it can be shown easily how they are connected with each other. We obtain:

$$\begin{aligned} 1 + z\tilde{F}_z(z,u) &= 1 + z\left(\frac{1-2Q}{Q(1-Q)}Q_z - \frac{1}{z}\right) = \frac{1-2Q}{Q(1-Q)}\frac{Q}{(1-2Q)(1-uQ)} \\ &= \frac{1}{(1-Q)(1-uQ)} = \tilde{M}(z,u). \end{aligned}$$

Thus, at the level of their coefficients, we obtain the following simple relation between the total number of tree and mapping parking functions extending Theorem 3.1.

Theorem 4.4. *For all $n \geq 1$ it holds that the total numbers $F_{n,m}$ and $M_{n,m}$ of (n,m)-tree parking functions and (n,m)-mapping parking functions, respectively, satisfy:*

$$M_{n,m} = n \cdot F_{n,m}.$$

This fact can be explained by extending the bijection presented in Theorem 3.4 to the general case $m \leq n$. This can be done as follows: Given a tree parking function (T, s) it is extended to a (n, n)-parking function $\tilde{s}$ for T in such a way that the last $(n-m)$ drivers all have their preferred parking spaces in those nodes that remained empty under s. Then the bijection ϕ is applied in order to obtain a (n, n) mapping parking function. Finally this mapping parking function is reduced to the first m drivers.

Using Proposition 4.3, extracting coefficients leads to the following explicit formulæ for the numbers $M_{n,m}$. Note that specializing $m = n$ restates Theorem 3.2.

Theorem 4.5. *The total number $M_{n,m}$ of (n, m)-mapping parking functions is, for $0 \leq m \leq n$ and $n \geq 1$, given as follows:*

$$M_{n,m} = \frac{(n-1)!m!n^{n-m}}{(n-m)!} \sum_{j=0}^{m} \binom{2m-n-j}{m-j} \frac{(2n)^j(n-j)}{j!}.$$

Proof. In view of the representation of $\tilde{M}(z, u)$ given in Proposition 4.3 containing the function $Q = Q(z, u)$, we make a change of variables in order to extract coefficients. Using the functional equation (15) and computing the derivative of Q w.r.t. z, an application of the Cauchy integral formula gives

$$\begin{aligned}
[z^n]\tilde{M}(z,u) &= \frac{1}{2\pi i} \oint \frac{\tilde{M}(z,u)}{z^{n+1}} dz = \frac{1}{2\pi i} \oint \frac{1}{z^{n+1}} \frac{1}{(1-Q)(1-uQ)} dz \\
&= \frac{1}{2\pi i} \oint \frac{e^{(uQ(1-Q)+2Q)(n+1)}}{(1-Q)(1-uQ)Q^{n+1}} \frac{(1-2Q)(1-uQ)}{e^{uQ(1-Q)+2Q}} dQ \\
&= [Q^n] \frac{e^{n(uQ(1-Q))}e^{2nQ}(1-2Q)}{1-Q}.
\end{aligned}$$

Further, for $0 \leq m \leq n$,

$$[z^n u^{n-m}]\tilde{M}(z,u) = \frac{n^{n-m}}{(n-m)!}[Q^m](1-Q)^{n-m-1}e^{2nQ}(1-2Q). \tag{20}$$

Using the negative generalization of binomial coefficients $\binom{-n}{k} = (-1)^k \binom{n-k-1}{k}$, we get

$$M_{n,m} = n!m![z^n u^{n-m}]\tilde{M}(z,u) = \frac{n!m!n^{n-m}}{(n-m)!}[Q^m](1-Q)^{n-m-1}e^{2nQ}(1-2Q)$$

$$= \frac{n!m!n^{n-m}}{(n-m)!} \sum_{j=0}^{m} \binom{n-m-1}{j} (-1)^j [Q^{m-j}] e^{2nQ}(1-2Q) \quad (21)$$

$$= \frac{n!m!n^{n-m}}{(n-m)!} \sum_{j=0}^{m} \binom{n-m-1}{j} (-1)^j \frac{2(n-m+j)(2n)^{m-j-1}}{(m-j)!}$$

$$= \frac{(n-1)!m!n^{n-m}}{(n-m)!} \sum_{j=0}^{m} \binom{2m-n-j}{m-j} \frac{(n-j)\,(2n)^j}{j!}.$$

□

4.3. Asymptotic considerations

The following question is of particular interest to us: How does the probability $p_{n,m} := M_{n,m}/n^{n+m}$ that a randomly chosen sequence of length m on the set $[n]$ is a parking function for a randomly chosen n-mapping swap from being equal to 1 (which is the case for $m = 1$) to being close to 0 (which is the case for $m = n$) when the ratio $\rho := m/n$ increases?

In order to get asymptotic results for $M_{n,m}$ we start with the representation (21), which can be written as

$$M_{n,m} = \frac{n!m!n^{n-m}}{(n-m)!} A_{n,m}, \quad (22)$$

with

$$A_{n,m} = [w^m](1-2w)e^{2nw}(1-w)^{n-m} \quad (23)$$
$$= \frac{1}{2\pi i} \int_{\Gamma} \frac{(1-2w)e^{2nw}(1-w)^{n-m-1}}{w^{m+1}} dw = \frac{1}{2\pi i} \int_{\Gamma} g(w) e^{nh(w)} dw,$$

where we choose as contour a suitable simple positively oriented closed curve around the origin. The functions $g(w)$ and $h(w)$ are given as follows

$$g(w) := \frac{1-2w}{(1-w)w} \quad \text{and} \quad h(w) := 2w + \left(1 - \frac{m}{n}\right) \log(1-w) - \frac{m}{n} \log w.$$

We will use the integral representation (23) of $A_{n,m}$ and apply saddle point techniques (see, e.g., [13, 7] for instructive expositions of this method). In the terminology of [7] the integral (23) has the form of a "large power integral" and saddle points of the relevant part $e^{nh(w)}$ of the integrand can thus be found as the zeros of the derivative $h'(w)$. The resulting equation

$$h'(w) = 2 - \left(1 - \frac{m}{n}\right) \frac{1}{1-w} - \frac{m}{n} \frac{1}{w} = 0$$

yields two solutions: $w_1 = \frac{m}{n}$ and $w_2 = \frac{1}{2}$. In our asymptotic analysis we will have to distinguish whether $w_1 < w_2$, $w_1 > w_2$ or $w_1 = w_2$. Our results can be summed up in the following theorem.

Theorem 4.6. *The total number $M_{n,m}$ of (n,m)-mapping parking functions is asymptotically, for $n \to \infty$, given as follows (where δ denotes an arbitrary small, but fixed, constant):*

$$M_{n,m} \sim \begin{cases} \frac{n^{n+m+\frac{1}{2}}\sqrt{n-2m}}{n-m}, & \text{for } 1 \le m \le (\frac{1}{2}-\delta)n, \\ \frac{\sqrt{2}\,3^{\frac{1}{6}}\,\Gamma(\frac{2}{3})n^{\frac{3n}{2}-\frac{1}{6}}}{\sqrt{\pi}}, & \text{for } m = \frac{n}{2}, \\ \frac{m!}{(n-m)!} \cdot \frac{n^{2n-m+\frac{3}{2}}2^{2m-n+1}}{(2m-n)^{\frac{5}{2}}}, & \text{for } (\frac{1}{2}+\delta)n \le m \le n. \end{cases}$$

The transient behaviour of the sequence $M_{n,m}$ for $m \sim \frac{n}{2}$ could be described via Airy functions as illustrated in [9].

Let us fix the ratio $\rho = m/n$. This ratio can be interpreted as a "load factor"–a term used in open addressing hashing. Then the asymptotic behaviour of the probabilities $p_{n,m} = p_{n,\rho n}$ follows immediately.

Corollary 4.7. *The probability $p_{n,m}$ that a randomly chosen pair (f,s), with f an n-mapping and s a sequence in $[n]^m$, represents a parking function is asymptotically, for $n \to \infty$ and $m = \rho n$ with $0 < \rho < 1$ fixed, given as follows:*

$$p_{n,m} \sim \begin{cases} C_{<}(\rho), & \text{for } 0 < \rho < \frac{1}{2}, \\ C_{1/2} \cdot n^{-1/6}, & \text{for } \rho = 1/2, \\ C_{>}(\rho) \cdot n^{-1} \cdot (D_{>}(\rho))^n, & \text{for } 1/2 < \rho < 1, \end{cases}$$

with

$$C_{<}(\rho) = \frac{\sqrt{1-2\rho}}{1-\rho}, \qquad C_{1/2} = \sqrt{\frac{6}{\pi}}\frac{\Gamma(2/3)}{3^{1/3}} \approx 1.298\ldots,$$
$$C_{>}(\rho) = 2 \cdot \sqrt{\frac{\rho}{(1-\rho)(2\rho-1)^5}}, \qquad D_{>}(\rho) = \left(\frac{4\rho}{e^2}\right)^{\rho} \frac{e}{2(1-\rho)^{1-\rho}}.$$

From Corollary 4.7 it follows that the limiting probability $L(\rho)$ that all drivers can park successfully for a load factor ρ, as depicted on the left-hand side in Figure 6, is given as follows :

$$L(\rho) := \lim_{n\to\infty} p_{n,\rho n} = \begin{cases} \frac{\sqrt{1-2\rho}}{1-\rho}, & \text{for } 0 \le \rho \le \frac{1}{2}, \\ 0, & \text{for } \frac{1}{2} \le \rho \le 1. \end{cases}$$

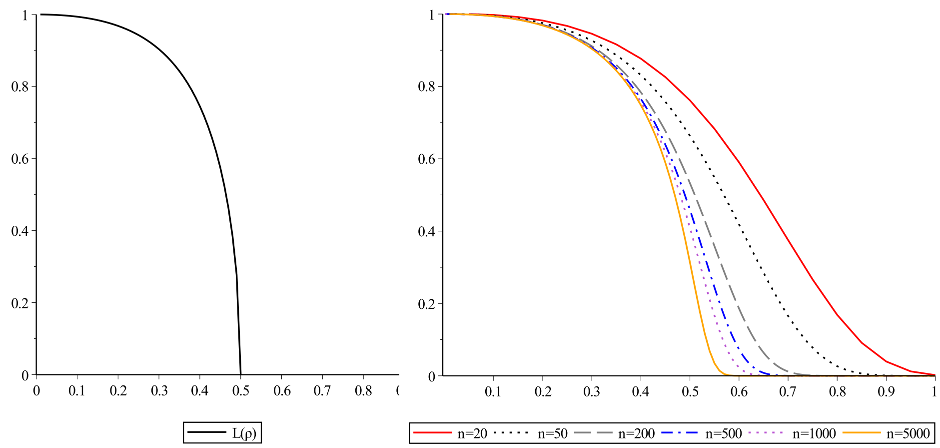

Figure 6: To the left: The limiting probability $L(\rho)$ that all drivers are able to park successfully in a mapping, for a load factor $0 \leq \rho \leq 1$. To the right: The exact probabilities $p_{n,\rho n}$ for $n = 20, 50, 200, 500, 1000, 5000$.

4.3.1. The region $\rho \leq \frac{1}{2} - \delta$

The geometry of the modulus of the integrand of (23) is easily described. There is a simple dominant saddle point at $w = w_1$, with one steepest descent/steepest ascent line following the real axis and another parallel to the imaginary axis. In Equation (23) we choose the contour Γ to be a circle centered at the origin and passing through the dominant saddle point w_1, i.e., it has radius $r = \rho$.

Using the parametrization $\Gamma = \{w = \rho e^{i\phi} : \phi \in [-\pi, \pi]\}$, we obtain from (23) the representation

$$A_{n,m} = \frac{1}{2\pi} \int_{-\pi}^{\pi} \rho e^{i\phi} g(\rho e^{i\phi}) e^{nh(\rho e^{i\phi})} d\phi. \tag{24}$$

Next we want to find a suitable splitting of the integral into the central approximation which should capture the main contribution of the integral and the remainder. That is, we need to choose a proper value $\phi_0 = \phi_0(n, m)$ to write the contour as $\Gamma = \Gamma_1 \cup \Gamma_2$, with $\Gamma_1 := \{w = \rho e^{i\phi} : \phi \in [-\phi_0, \phi_0]\}$ and $\Gamma_2 := \{w = \rho e^{i\phi} : \phi \in [-\pi, -\phi_0] \cup [\phi_0, \pi]\}$ yielding the representation $A_{n,m} = I_{n,m}^{(1)} + I_{n,m}^{(2)}$, such that $I_{n,m}^{(2)} = o\left(I_{n,m}^{(1)}\right)$, where $I_{n,m}^{(1)}$ is the integral over Γ_1 and $I_{n,m}^{(2)}$ the one over Γ_2. To do this we consider the local expansion of the integral around $\phi = 0$:

$$\rho e^{i\phi} g(\rho e^{i\phi}) = \frac{1 - \frac{2m}{n} e^{i\phi}}{1 - \frac{m}{n} e^{i\phi}} = \frac{1 - \frac{2m}{n}}{1 - \frac{m}{n}} \cdot \left(1 + \mathcal{O}\left(\frac{m\phi}{n}\right)\right),$$

$$
\begin{aligned}
nh(\rho e^{i\phi}) =& n\left(\frac{2m}{n}e^{i\phi}+\left(1-\frac{m}{n}\right)\log\left(1-\frac{m}{n}e^{i\phi}\right)-\frac{m}{n}\log\left(\frac{m}{n}e^{i\phi}\right)\right)\\
=& 2m+(n-m)\log\left(1-\frac{m}{n}\right)-m\log\left(\frac{m}{n}\right)\\
& -\frac{m(n-2m)}{2(n-m)}\phi^2+\mathcal{O}(m\phi^3),
\end{aligned}
$$

yielding

$$
\begin{aligned}
\rho e^{i\phi}g(\rho e^{i\phi})e^{nh(\rho e^{i\phi})} = & \left(1-\frac{2m}{n}\right)\left(\frac{n}{m}\right)^m e^{2m}\left(1-\frac{m}{n}\right)^{n-m-1} e^{-\left(\frac{m(n-2m)}{2(n-m)}\right)\phi^2}\cdot\\
& \cdot\left(1+\mathcal{O}\left(m\phi^3\right)+\mathcal{O}\left(\frac{m\phi}{n}\right)\right).
\end{aligned}
$$

From the latter expansion we obtain that we need to choose ϕ_0 such that $m\phi_0^2\to\infty$ (then the central approximation contains the main contributions) and $m\phi_0^3\to 0$ (then the remainder term is asymptotically negligible). E.g., we may choose $\phi_0=m^{-\frac{1}{2}+\frac{\epsilon}{3}}$, for a constant $0<\epsilon<\frac{1}{2}$. With such a choice of ϕ_0 and the substitution $\phi=\frac{t}{\sqrt{m}}$ we obtain the following asymptotic expansion for $I_{n,m}^{(1)}$:

$$
\begin{aligned}
I_{n,m}^{(1)} =& \frac{e^{2m}}{2\pi}\left(1-\frac{2m}{n}\right)\left(\frac{n}{m}\right)^m\left(1-\frac{m}{n}\right)^{n-m-1}\frac{1}{\sqrt{m}}\cdot\int_{-m^{\frac{\epsilon}{3}}}^{m^{\frac{\epsilon}{3}}} e^{-\left(\frac{n-2m}{2(n-m)}\right)t^2}dt\cdot\\
& \cdot\left(1+\mathcal{O}\left(m^{-\frac{1}{2}+\epsilon}\right)\right).
\end{aligned}
$$

For the tail completion we use that $\int_c^\infty e^{-\alpha t^2}dt=\mathcal{O}\left(e^{-\alpha c^2}\right)$, for $c>0$ and $\alpha>0$. Thus we obtain

$$
\int_{m^{\frac{\epsilon}{3}}}^{\infty} e^{-\left(\frac{n-2m}{2(n-m)}\right)t^2}dt=\mathcal{O}\left(e^{-\left(\frac{n-2m}{2(n-m)}\right)m^{\frac{2\epsilon}{3}}}\right),
$$

which yields a subexponentially small and thus negligible error term. Using this, we may proceed in the asymptotic evaluation of $I_{n,m}^{(1)}$ and get

$$
\begin{aligned}
I_{n,m}^{(1)} =& \frac{e^{2m}}{2\pi\sqrt{m}}\left(1-\frac{2m}{n}\right)\left(\frac{n}{m}\right)^m\left(1-\frac{m}{n}\right)^{n-m-1}\\
& \cdot\int_{-\infty}^{\infty} e^{-\left(\frac{n-2m}{2(n-m)}\right)t^2}dt\cdot\left(1+\mathcal{O}\left(m^{-\frac{1}{2}+\epsilon}\right)\right)\\
=& \frac{e^{2m}(n-m)^{n-m-\frac{1}{2}}\sqrt{n-2m}}{\sqrt{2\pi}\,m^{m+\frac{1}{2}}n^{n-2m}}\cdot\left(1+\mathcal{O}\left(m^{-\frac{1}{2}+\epsilon}\right)\right).
\end{aligned}
$$

Next we consider the remainder integral $I^{(2)}_{n,m}$. To estimate the integrand we use the obvious bounds

$$\left|1-\frac{2m}{n}e^{i\phi}\right| \le 1+\frac{2m}{n} \quad \text{and} \quad \frac{1}{\left|1-\frac{m}{n}e^{i\phi}\right|} \le \frac{1}{1-\frac{m}{n}},$$

as well as the following:

$$\begin{aligned}&\left|e^{n\left(2\frac{m}{n}e^{i\phi}+(1-\frac{m}{n})\log(1-\frac{m}{n}e^{i\phi})-\frac{m}{n}\log(\frac{m}{n})-\frac{m}{n}i\phi\right)}\right|\\ =&\left(\frac{n}{m}\right)^m e^{n\left(2\rho\cos\phi+\frac{1-\rho}{2}\log(1-2\rho\cos\phi+\rho^2)\right)}.\end{aligned}$$

This yields

$$\left|I^{(2)}_{n,m}\right| \le \frac{1}{2\pi}\frac{(1-\frac{2m}{n})(1+\frac{2m}{n})}{1-\frac{m}{n}}\cdot\left(\frac{n}{m}\right)^m\cdot\int_{\Gamma_2} e^{n\left(2\rho\cos\phi+\frac{1-\rho}{2}\log(1-2\rho\cos\phi+\rho^2)\right)}d\phi.$$

Using standard calculus it can be showed that amongst all points of the contour Γ_2 the integrand reaches its maximum at $\phi=\phi_0$. Thus, we obtain

$$\begin{aligned}\left|I^{(2)}_{n,m}\right| \le&\frac{(1-\frac{2m}{n})(1+\frac{2m}{n})}{1-\frac{m}{n}}\cdot\left(\frac{n}{m}\right)^m\cdot e^{2m\cos\phi_0+\frac{n-m}{2}\log\left(1-\frac{2m}{n}\cos\phi_0+(\frac{m}{n})^2\right)}\\ \le&2\cdot\left(\frac{n}{m}\right)^m\cdot e^{2m\cos\phi_0+\frac{n-m}{2}\log\left(1-\frac{2m}{n}\cos\phi_0+(\frac{m}{n})^2\right)}\\ =&2\left(\frac{n}{m}\right)^m e^{2m}\left(1-\frac{m}{n}\right)^{n-m}\cdot e^{2m(\cos\phi_0-1)+\frac{n-m}{2}\log\left(1-\frac{\frac{2m}{n}(\cos\phi_0-1)}{(1-\frac{m}{n})^2}\right)}.\end{aligned}$$

Using the estimates $\log(1-x)\le -x$ for $x<1$ and $\cos x-1\le-\frac{x^2}{6}$ for $x\in[-\pi,\pi]$ we obtain that:

$$e^{2m(\cos\phi_0-1)+\frac{n-m}{2}\log\left(1-\frac{\frac{2m}{n}(\cos\phi_0-1)}{(1-\frac{m}{n})^2}\right)} \le e^{-\frac{1}{6}\left(2-\frac{1}{1-\frac{m}{n}}\right)m^{\frac{2\epsilon}{3}}}.$$

Thus we obtain

$$|I^{(2)}_{n,m}| = |I^{(1)}_{n,m}|\cdot\mathcal{O}\left(\sqrt{m}\,e^{-cm^{\frac{2\epsilon}{3}}}\right), \quad \text{with } c=\frac{1}{6}\left(2-\frac{1}{1-\frac{m}{n}}\right),$$

i.e., $I^{(2)}_{n,m}$ is subexponentially small compared to $I^{(1)}_{n,m}$.

Combining these results we get

$$A_{n,m}=\frac{(n-m)^{n-m-\frac{1}{2}}\sqrt{n-2m}\,e^{2m}}{\sqrt{2\pi}\,m^{m+\frac{1}{2}}n^{n-2m}}\cdot\left(1+\mathcal{O}\left(m^{-\frac{1}{2}+\epsilon}\right)\right)$$

and, by using (22) and applying Stirling's formula,

$$M_{n,m} = \frac{n!m!n^{n-m}(n-m)^{n-m-\frac{1}{2}}\sqrt{n-2m}\,e^{2m}}{\sqrt{2\pi}\,(n-m)!m^{m+\frac{1}{2}}n^{n-2m}} \cdot \left(1 + \mathcal{O}\left(m^{-\frac{1}{2}+\epsilon}\right)\right)$$
$$= \frac{n^{n+m}\sqrt{1-\frac{2m}{n}}}{1-\frac{m}{n}} \cdot \left(1 + \mathcal{O}\left(m^{-\frac{1}{2}+\epsilon}\right)\right). \tag{25}$$

Note that according to the remainder term in (25) we have only shown the required result for $m \to \infty$. However, again starting with (24), we can easily show a refined bound on the error term for small m. Namely, we may write the integral as follows:

$$A_{n,m} = \frac{1}{2\pi}\left(\frac{n}{m}\right)^m \cdot \int_{-\pi}^{\pi} \frac{e^{2me^{i\phi}}(1-\frac{m}{n}e^{i\phi})^{n-m-1}(1-\frac{2m}{n}e^{i\phi})}{e^{im\phi}} d\phi,$$

and use for $m = o(\sqrt{n})$ the expansions

$$\left(1-\frac{m}{n}e^{i\phi}\right)^{n-m-1} = e^{-me^{i\phi}} \cdot \left(1 + \mathcal{O}\left(\frac{m^2}{n}\right)\right), \quad 1 - \frac{2m}{n}e^{i\phi} = 1 + \mathcal{O}\left(\frac{m}{n}\right),$$

which gives

$$A_{n,m} = \frac{1}{2\pi}\left(\frac{n}{m}\right)^m \cdot \int_{-\pi}^{\pi} \frac{e^{me^{i\phi}}}{e^{im\phi}} d\phi \cdot \left(1 + \mathcal{O}\left(\frac{m^2}{n}\right)\right).$$

4.3.2. The region $\rho \geq \frac{1}{2} + \delta$

This region can be treated analogously to the previous one by choosing the contour Γ in (23) to be a circle centered at the origin and with radius $r = \frac{1}{2}$. The contour may again be split into two parts, $\Gamma = \Gamma_1 \cup \Gamma_2$, with $\Gamma_1 = \{w = \frac{1}{2}e^{i\phi} : \phi \in [-\phi_0, \phi_0]\}$ and $\Gamma_2 = \{w = \frac{1}{2}e^{i\phi} : \phi \in [-\pi, -\phi_0] \cup [\phi_0, \pi]\}$ and where $\phi_0 = n^{-\frac{1}{2}+\epsilon}$, with $0 < \epsilon < \frac{1}{6}$. The further calculations and asymptotic estimations are not detailed here.

4.3.3. The monkey saddle for $\rho = 1/2$

For $\rho = \frac{m}{n} = \frac{1}{2}$, the situation is slightly different to the previous regions since the two otherwise distinct saddle points coalesce to a unique double saddle point. The difference in the geometry of the surface is that there are now three steepest descent lines and three steepest ascent lines departing from the saddle point at angles $0, 2\pi/3$ and $-2\pi/3$.

Thus, we may choose as integration contour two line segments joining the point $w = \frac{1}{2}$ with the imaginary axis at an angle of $-2\pi/3$ and $2\pi/3$, respectively, as well as a half circle centered at the origin and joining the two line segments. This yields $\Gamma = \Gamma_1 \cup \Gamma_2 \cup \Gamma_3$ and $A_{n,m} = I^{(1)}_{n,m} + I^{(2)}_{n,m} + I^{(3)}_{n,m}$ for the corresponding integrals. We first treat

$$I^{(1)}_{n,m} = -\frac{e^{-\frac{2\pi i}{3}}}{2\pi i} \int_0^1 g\left(\frac{1}{2} + e^{-\frac{2\pi i}{3}} t\right) e^{nh\left(\frac{1}{2}+e^{-\frac{2\pi i}{3}} t\right)} dt.$$

In order to find a suitable choice t_0 for splitting the integral we consider the expansion of the integrand around $t = 0$:

$$g\left(\frac{1}{2} + e^{-\frac{2\pi i}{3}} t\right) e^{nh\left(\frac{1}{2}+e^{-\frac{2\pi i}{3}} t\right)} = -8e^n e^{-\frac{2\pi i}{3}} t e^{-\frac{8}{3}nt^3} \cdot \left(1 + \mathcal{O}(t^2) + \mathcal{O}(nt^5)\right).$$

Thus we obtain the restrictions $nt_0^3 \to \infty$ and $nt_0^5 \to 0$ which are, e.g., satisfied when choosing $t_0 = n^{-\frac{1}{4}}$. This splitting yields $I^{(1)}_{n,m} = I^{(1,1)}_{n,m} + I^{(1,2)}_{n,m}$, for the integration paths $t \in [0, t_0]$ and $t \in [t_0, 1]$, respectively.

Using the local expansion of the integrand as well as the before-mentioned choice for t_0, the central approximation $I^{(1,1)}_{n,m}$ gives

$$I^{(1,1)}_{n,m} = \frac{4e^n e^{-\frac{4\pi i}{3}}}{\pi i} \int_0^\infty t e^{-\frac{8}{3}nt^3} dt \cdot \left(1 + \mathcal{O}(n^{-\frac{1}{4}})\right),$$

since one can show easily that completing the integral only yields a subexponentially small error term. Moreover, also the remainder $I^{(1,2)}_{n,m}$ only yields a subexponentially small error term compared to $I^{(1,1)}_{n,m}$. The integral $I^{(2)}_{n,m}$ can be treated in an analogous manner:

$$I^{(2)}_{n,m} = -\frac{4e^n e^{\frac{4\pi i}{3}}}{\pi i} \int_0^\infty t e^{-\frac{8}{3}nt^3} dt \cdot \left(1 + \mathcal{O}(n^{-\frac{1}{4}})\right).$$

Moreover, one can show that the contribution of

$$I^{(3)}_{n,m} = \frac{1}{2\pi i} \int_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \frac{\sqrt{3}}{2} i e^{it} g\left(\frac{\sqrt{3}}{2} e^{it}\right) e^{nh\left(\frac{\sqrt{3}}{2} e^{it}\right)} dt$$

is asymptotically negligible compared to $I^{(1)}_{n,m}$ and $I^{(2)}_{n,m}$.

Collecting the contributions and evaluating the integral yields

$$A_{n,m} \sim \frac{4e^n}{\pi i}\left(e^{-\frac{4\pi i}{3}} - e^{\frac{4\pi i}{3}}\right) \cdot \int_0^\infty t e^{-\frac{8}{3}nt^3} dt = \frac{4\sqrt{3}\, e^n}{\pi} \frac{\Gamma(\frac{2}{3})}{4\sqrt[3]{3} n^{2/3}},$$

and thus by using (22):

$$M_{n,m} \sim \frac{\sqrt{2}\, 3^{\frac{1}{6}} \Gamma(\frac{2}{3}) n^{\frac{3n}{2}}}{\sqrt{\pi}\, n^{\frac{1}{6}}}.$$

5. Conclusion

This paper constitutes the first treatment of parking functions for mappings. Let us mention some possible further research directions.

1. Given a mapping f or a tree T, we obtained general, but simple bounds for the number of tree and mapping parking functions $S(f, m)$ and $S(T, m)$. For some simple classes of trees, e.g., for "chain-like" trees with only few branchings it is possible to obtain explicit formulæ. Is it possible in general to give some "simple characterization" of the numbers $S(f, m)$ and $S(T, m)$?
2. Let us denote by X_n the random variable measuring the number of parking functions s with n drivers for a randomly chosen n-mapping. Then, by Corollary 3.3, the expected value of X_n is given by $\mathbb{E}(X_n) = M_n/n^n \sim \sqrt{2\pi}\, 2^{n+1} n^{n-\frac{1}{2}} e^{-n}$. However, with the approach presented here, it seems that we are not able to obtain higher moments or other results on the distribution of X_n.
3. As for ordinary parking functions one could analyse important quantities for mapping parking functions. For instance the so-called "total displacement" (which is of particular interest in problems related to hashing algorithms, see [14, 15]), i.e., the total driving distance of the drivers, or "individual displacements" (the driving distance of the k-th driver, see [16, 17]) seem to lead to interesting questions. Moreover, the "sums of parking functions" as studied in [18] could be worthwhile treating as well.
4. A refinement of parking functions can be obtained by studying what has been called "defective parking functions" in [19], or "overflow" in [20], i.e., pairs (f, s), such that exactly k drivers are unsuccessful. Preliminary studies indicate that the approach presented is suitable to obtain results in this direction as well.
5. Besides Cayley trees which are a special case of mappings, one could also study the total number of parking functions for other important tree families as, e.g., labelled binary trees, labelled ordered trees or for increasing tree families (the labels along all leaf-to-root-paths form an increasing sequence, see, e.g., [21, 22]).

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