Fundamentals of Mathematics I

VO B01.05

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Lecture given at

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Content

Outline

Defintion

A **set** is a well defined collection of distinct objects. The objects that make up a set are called **elements** or **members**.

- The elements can be anything: numbers, people, letters of the alphabet, other sets, and so on.
- There are two ways of defining a set:
  - by describing its elements; e.g. M is the set of all inhabitants of Krems
  - by listing the elements; e.g. $M := \{1, 2, 3, 4\}$
- There is no order on the elements of a set; e.g. $\{1, 2, 3, 4\} = \{2, 1, 4, 3\}$
- The set that contains no elements is called the **empty set** and is denoted by $\emptyset$.
- Two sets are the same if they contain the same elements.

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Notation

We write $x \in M$ if x is an element of M and $x \notin M$ if x is not an element of M.

Example: $4 \in \{1, 2, 3, 4\}$ but $5 \notin \{1, 2, 3, 4\}$

Definition

If every element of set A is also an element of set B, then A is said to be a **subset** of B, written $A \subseteq B$. One also says "A is contained in B".

Be careful to make the difference between elements and subsets!

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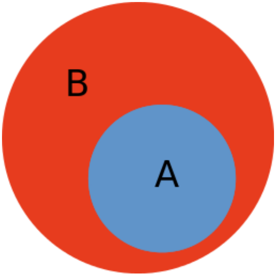
Examples:

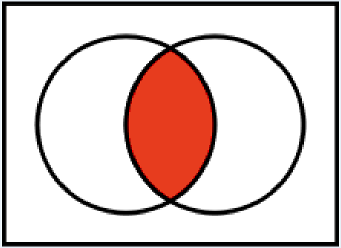
- $\{4\} \subseteq \{1,2,3,4\}$
  whereas $4 \subseteq \{1,2,3,4\}$ does not make sense.
- For every set M it holds that $\emptyset \subseteq M$ and $M \subseteq M$.
- The subsets of $M := \{\bigstar, \bullet\}$ are the following: $\emptyset$, $\{\bigstar\}$, $\{\bullet\}$ and M itself.
- In order to depict sets and there subsets one can use so-called Venn diagrams: $A \subseteq B$.



Given two sets A and B, we define the following:

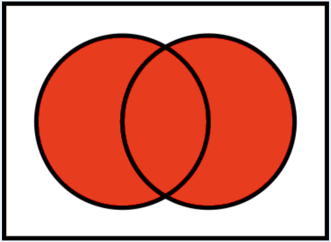
Definition

The **intersection** of A and B, denoted by $A \cap B$, is the set of elements contained in A <u>and</u> in B.



Definition

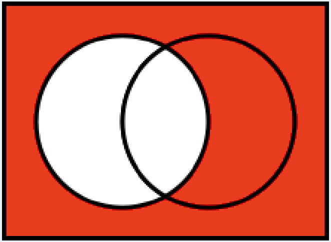
The **union** of A and B, denoted by $A \cup B$, is the set of elements contained in A <u>or</u> in B.



In certain settings all sets under discussion are considered to be subsets of a given universal set U. We can then define the following:

Definition

The complement of a set A (within U), denoted by A^c, is the set of all elements not contained in A.



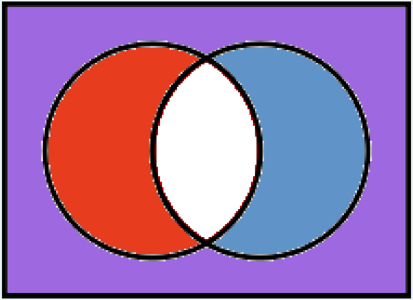
Examples:

- Let $A := \{1,2,3,4\}$ and $B := \{1,3,7,8\}$
  Then $A \cap B = \{1,3\}$ and $A \cup B = \{1,2,3,4,7,8\}$
- For every set M it holds that $M \cap M = M$ and $M \cup M = M$.
  Moreover $M \cap \emptyset = \emptyset$ and $M \cup \emptyset = M$.
- Let $U := \{1,2,3,4,5,6,7,8\}$. Then $A^c = \{5,6,7,8\}$ and $B^c = \{2,4,5,6\}$.
- For every set M $(\subseteq U)$ it holds that $M \cap M^c = \emptyset$. One says that M and M^c are **disjoint**.
- Let $C := \{5,6\}$. Then $A \cap C = \emptyset$; A and C are disjoint.
- For every set M it holds that $(M^c)^c = M$.

De Morgan's rules

For two sets A and B the following holds:

$$(A \cap B)^c = A^c \cup B^c \quad \text{and} \quad (A \cup B)^c = A^c \cap B^c$$



In the following we will encounter numbers that belong to certain special sets:

- The **natural numbers** (or positive integers): 1, 2, 3, 4, and so on. This set is denoted by $\mathbb{N}$.
- The **integers** (the positive and the negative integers and the element 0): ...,-4, -3, -2, -1, 0, 1, 2, 3, ... This set is denoted by $\mathbb{Z}$.

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- The **rational numbers** are fractions, i.e. that can be written as $\frac{m}{n}$ where m and n are integers. These are numbers such as $1/2, -1/3, 2/7, \ldots$. This set is denoted by $\mathbb{Q}$.

- The **real numbers** represent a position along a continuous line. These are the rational numbers together with the irrational numbers, such as $\sqrt{2}, \pi, e, \ldots$. This set is denoted by $\mathbb{R}$.
  We use the following notation for **intervals**:
  $[a, b]$ ... all reals x with $a \leq x \leq b$
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For these sets the following holds:

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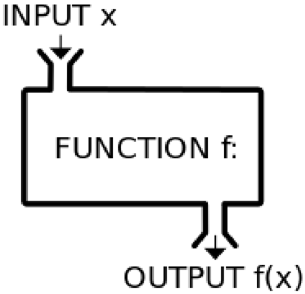
What is a function?

Definition

A **function** $f : X \to Y$ is a rule that associates to every element in X a unique element of Y.



X is called the **domain** of f, Y its **range** or **codomain**. x is the **variable** or **argument** and $f(x) = y$ is the **value**. One writes $x \mapsto f(x)$ and says "x is mapped to $f(x)$"

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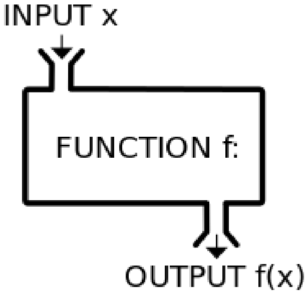

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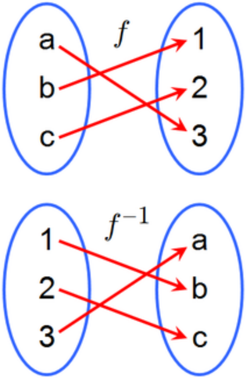
Examples

- X is the set of all inhabitants of Krems,
  Y is the set of positive integers $\mathbb{N}$
  f is the function that associates to each person its age in days
- $X = [0, 100]$ and $Y = \mathbb{N}$ and the function f describes the growth of a bacteria population over time during some experiment. The variable $x \in X$ thus represents time.
- $f(x) = \pm\sqrt{x}$ for $x \geq 0 \in \mathbb{R}$ is not a function, since it assigns to each positive real number x two values: the (positive) square root of x, and $-\sqrt{x}$.

The inverse of a function

If the function f maps x to y, the inverse function f^{-1} maps y to x.



Be careful: The inverse does not always exist on its entire domain! If two elements x_1 and x_2 are mapped to the same value y, the inverse cannot be determined uniquely!

Example: The inverse of $f(x) = x^2$ on the positive real numbers is $\sqrt{x}$ and is $-\sqrt{x}$ on the negative real numbers. On its entire domain, f does not have an inverse!

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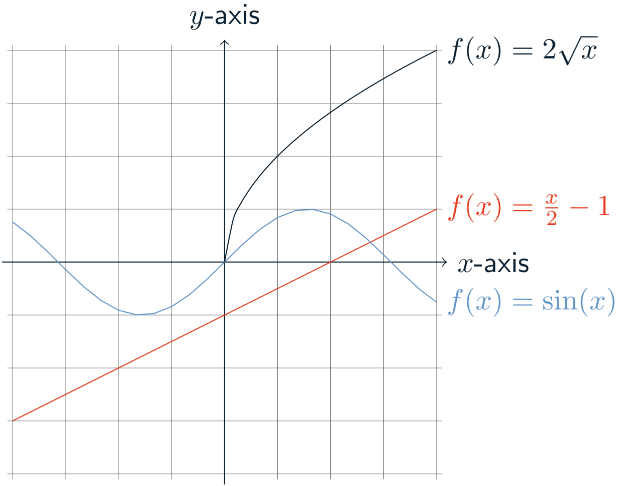
A convenient way to represent a function defined on some real interval or on the integers is to draw its **plot** or **graph**:



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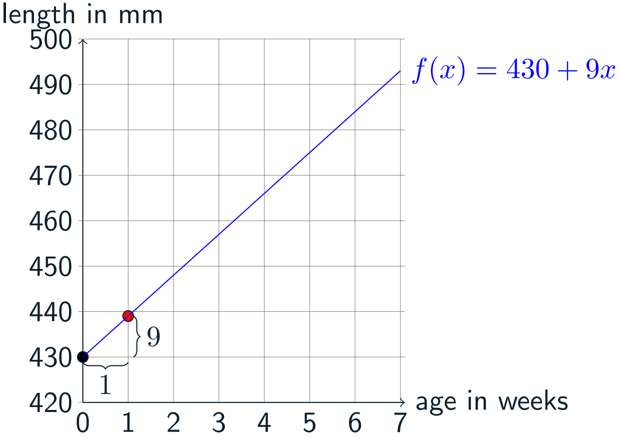

A linear function is a special form of a polynomial function, it is a polynomial of degree 1.

Definition

A **polynomial** is a function defined on $\mathbb{R}$ of the following type

$$f(x) = a_0 + a_1 x + a_2 x^2 + \ldots + a_m x^m,$$

where the a_i are real numbers. The number m is a positive integer and is called the **degree** of f.
A number $x_0 \in \mathbb{R}$ is called a **zero** of f if it holds that $f(x_0) = 0$.

A polynomial of degree m has at most m zeros. In the complex numbers, a polynomial of degree m has exactly m zeros.

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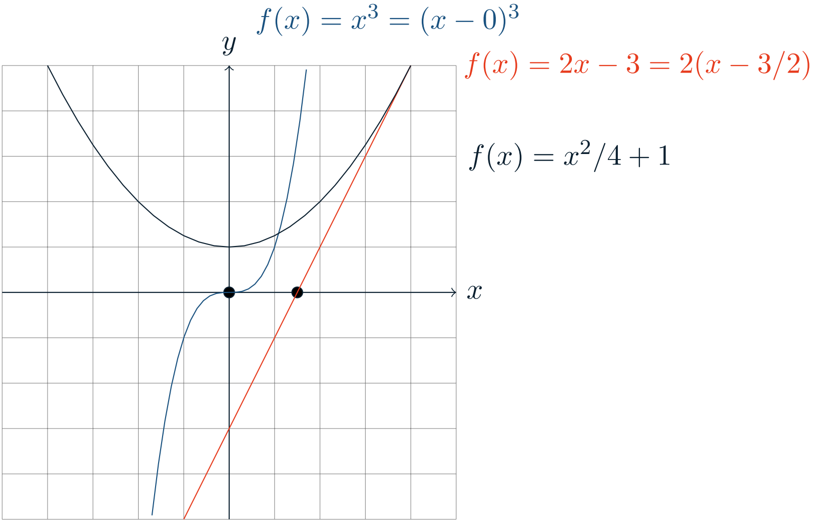

Solving quadratic equations

The zeros of a **quadratic function**, that is a polynomial of degree 2, can be found using the following formula:

$$a + bx + cx^2 = 0 \Leftrightarrow x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2c}$$

Thus the solutions are real whenever $b^2 - 4ac \geq 0$ and when $b^2 - 4ac = 0$ there are two coinciding zeros.

Exponential function

The **exponential** function $f(x) = e^x$ is often used to model the growth or decay of some population. **Euler's constant** e is approximately 2.718. The formula used for exponential growth or decay is $N = N_0 \cdot e^{k \cdot t}$, where N is the changing quantity, t is time, N_0 is its value at $t = 0$ and k is the growth constant.

Example: The growth of some bacteria population placed out onto agar is given by the following formula:

$$N = 100 \cdot e^{t/5},$$

where t is in hours.

Question: How long does it take until the population doubles?

t must satisfy $e^{t/5} = 2$, this is the case for $t \approx 3.47$, so after roughly 3.5 hours.

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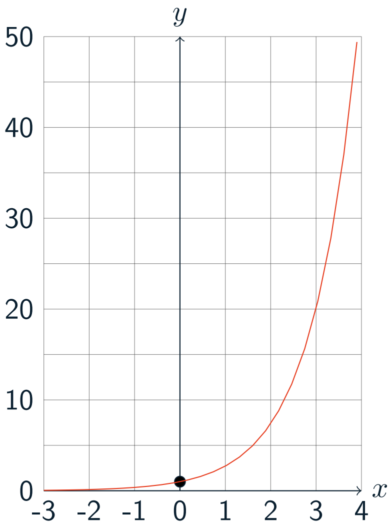
Properties of the e-function



- $e^0 = \exp(0) = 1$
- $e^x > 0$ for all $x \in \mathbb{R}$
- $e^{x+y} = e^x \cdot e^y$
- $e^{-x} = \frac{1}{e^x}$
- $e^{x \cdot y} = (e^x)^y = (e^y)^x$

Logarithms

The **natural logarithm** $\ln(x)$ is the inverse function of the exponential function and is defined on the positive reals. Thus the following holds:

$$e^x = y \Leftrightarrow x = \ln(y).$$

We can also define logarithms for other bases:
The **logarithm to the base** a is the inverse of the function a^x and is denoted by $\log_a(x)$. The following holds:

$$\log_a(x) = \frac{\log_e(x)}{\log_e(a)} = \frac{\ln(x)}{\ln(a)}.$$

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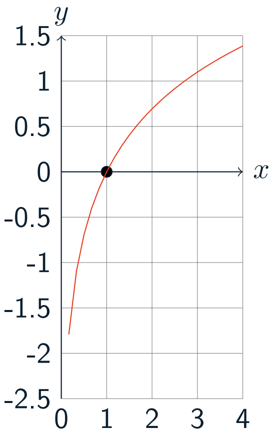
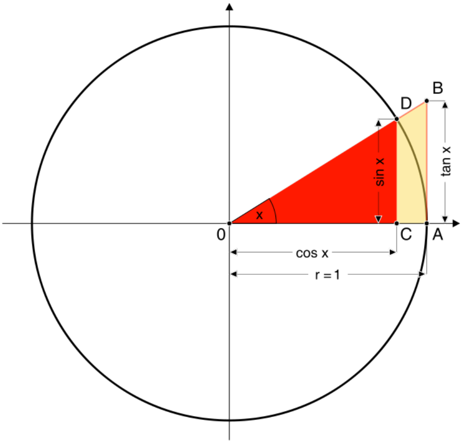


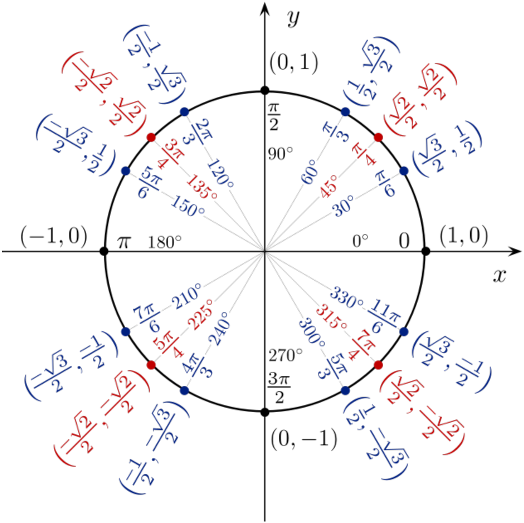
- $\ln(1) = 0$
- $\ln(x \cdot y) = \ln(x) + \ln(y)$
- $\ln\left(\frac{1}{x}\right) = -\ln(x)$
- $\ln\left(x^y\right) = y\ln(x)$

Cosine, sine, and tangent



- $\cos(x) = \frac{\text{adjacent}}{\text{hypotenuse}}$
- $\sin(x) = \frac{\text{opposite}}{\text{hypotenuse}}$
- $\tan(x) = \frac{\text{opposite}}{\text{adjacent}} = \frac{\sin(x)}{\cos(x)}$

Some important values



Trigonometric functions as real functions

As real functions, the cosine, sine and tangent are $360°$-respectively 2π-periodic, i.e. $\sin(x + 2\pi) = \sin(x)$.

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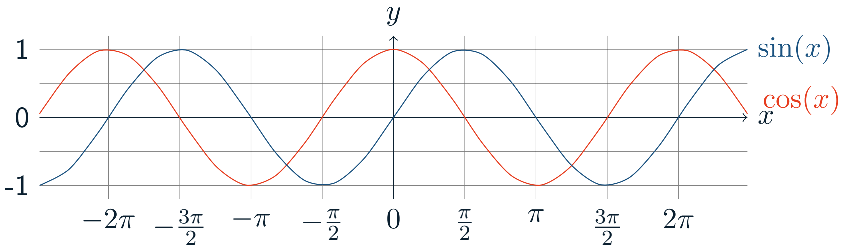
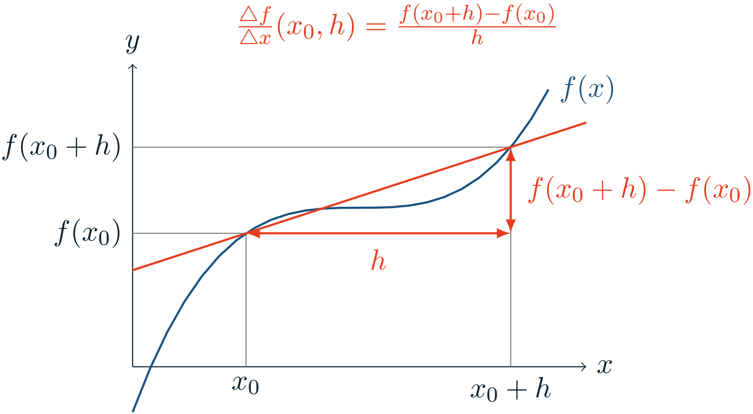


Outline

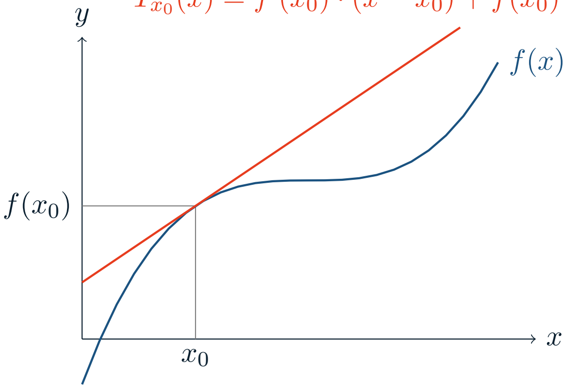
Differential quotient



Tangent

Tangent to f at the point x_0

$$T_{x_0}(x) = f'(x_0) \cdot (x - x_0) + f(x_0)$$



Derivatives of some simple functions

For the derivative of a function $f(x)$ we write $f'(x)$ or $\frac{df}{dx}$.

- $f(x) = c \qquad f'(x) = 0$,
  where $c \in \mathbb{R}$ is some constant
- $f(x) = x^k \qquad f'(x) = k \cdot x^{k-1}$,
  where $k \in \mathbb{Z}$
- $f(x) = \sqrt{x} = x^{1/2} \qquad f'(x) = \frac{1}{2\sqrt{x}}$
- $f(x) = \ln(x) \qquad f'(x) = \frac{1}{x}$
- $f(x) = e^x \qquad f'(x) = e^x$
- $f(x) = \cos(x) \qquad f'(x) = -\sin(x)$
- $f(x) = \sin(x) \qquad f'(x) = \cos(x)$

Rules

- Constant factor $(c \cdot f(x))' = c \cdot f'(x)$, where $c \in \mathbb{R}$ is some constant
- Sum $(f(x) + g(x))' = f'(x) + g'(x)$
- Product $(f(x) \cdot g(x))' = f'(x) \cdot g(x) + f(x) \cdot g'(x)$
- Quotient $\left(\frac{f(x)}{g(x)}\right)' = \frac{f'(x) \cdot g(x) - f(x) \cdot g'(x)}{(g(x))^2}$
- Chain rule $(f(g(x)))' = f'(g(x)) \cdot g'(x)$

Examples

- What is the derivative of $f(x) = (2-x)^2 + 5\ln(x)$?
  1. Apply the **sum** rule: $f'(x) = \left((2-x)^2\right)' + (5\ln(x))'$
  2. Apply the **chain** rule on the left part: $\left((2-x)^2\right)' = -2(2-x)$
  3. Apply the **constant factor** rule on the right part: $(5\ln(x))' = 5 \cdot (\ln(x))' = 5 \cdot 1/x$
  4. Put everything together: $f'(x) = 2x + \frac{5}{x} - 4$

Examples

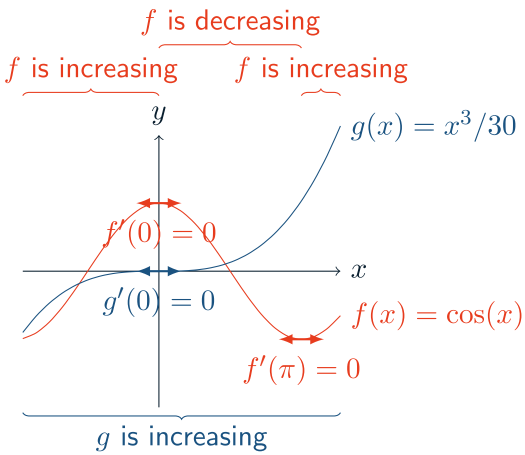
- What is the derivative of $f(x) = \tan(2x)$?
  1. Express $\tan(x)$ using $\cos(x)$ and $\sin(x)$: $f(x) = \frac{\sin(2x)}{\cos(2x)}$
  2. Apply the **quotient** rule:
     $\left(\frac{\sin(2x)}{\cos(2x)}\right)' = \frac{(\sin(2x))' \cdot \cos(2x) - \sin(2x) \cdot (\cos(2x))'}{(\cos(2x))^2}$
  3. Apply the **chain** rule to $\sin(2x)$ and $\cos(2x)$:
     $(\sin(2x))' = \cos(2x) \cdot 2$, $(\cos(2x))' = -\sin(2x) \cdot 2$
  4. Put everything together:

$$
\begin{aligned}
f'(x) &= \frac{2\cos(2x)\cos(2x) + 2\sin(2x)\sin(2x)}{(\cos(2x))^2} \\
&= 2 \cdot \left(1 + \left(\frac{\sin(2x)}{\cos(2x)}\right)^2\right) = 2 \cdot \left(1 + \tan(2x)^2\right) \\
&= \frac{2}{\cos(2x)^2}
\end{aligned}
$$

What does the derivative tell us about a function?

- If $f'(x_0) = 0$ the point x_0 is a **stationary point**: it is either a *saddle point* or a **local extremum**, that is a **local maximum** or a **local minimum**.
  We will see how to tell the difference in a moment.
- $f'(x) > 0$ on some interval $[a, b]$, the function f is **strictly increasing** on $[a, b]$. This means the following: if $x_1 < x_2 \in [a, b]$, then $f(x_1) < f(x_2)$
- $f'(x) < 0$ on some interval $[a, b]$, the function f is **strictly decreasing** on $[a, b]$.

Examples


f is decreasing
f is increasing
f is increasing
y
$g(x) = x^3/30$
$f'(0) = 0$
x
$g'(0) = 0$
$f(x) = \cos(x)$
$f'(\pi) = 0$
g is increasing

$f'(x_0) = 0$

We distinguish three cases:

- $f''(x_0) > 0$: then x_0 is a **local minimum**
  Example: $f(x) = \cos(x)$ and $x_0 = \pi$
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- $f''(x_0) = 0$ and $f'''(x_0) \neq 0$: then x_0 is a **saddle point**
  Example: $f(x) = x^3$ and $x_0 = 0$
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Example: Logistic growth

When life scientists need to model a behaviour in which the dependent variable initially increases, but soon levels off and then drops back down to its starting value, they speak of **logistic growth**. Here's an example that you have probably already experienced yourself: the harder you study for an exam, the better you will do in it - but every extra hour of study results in a smaller pay-off and beyond a certain point further study can be harmful: if you overdo it and study all night, you will be so exhausted in the morning that you fail the exam!

Such a behaviour can be modelled as follows:

$$f(t) = at \cdot \left(1 - \frac{t}{k}\right) = -\frac{a}{k}t^2 + at,$$

where t represents the hours of study and $f(t)$ the performance achieved. How do the constants a and k have to be chosen?

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- Where are the zeros of f?
- Where is f increasing, where decreasing?
- When should you definitely stop studying?

Now, let's draw the plot:

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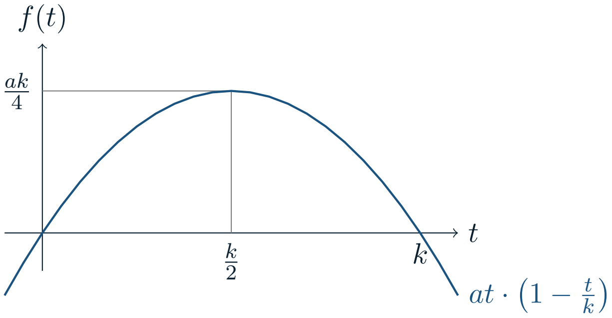

Applications

You can use differential calculus to:

- **Learn more** about a function: where does it have minima and maxima? Where are inflection points?
- If a function $f(t)$ represents the distance an object has travelled over time, then its first derivative $f'(t)$ is the **speed** at the moment t and the second derivative $f''(t)$ is the **acceleration** at the moment t.
- Solve **optimization problems**: In many applications, one is interested in maximizing or minimizing a certain parameter, for instance the costs.
- **Approximate** the growth a function within a small interval: If x is close to x_0 then $f(x) \approx f(x_0) + f'(x_0) \cdot (x - x_0)$.

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Outline

Integration is the reverse process to differentiation

We learnt that the differential of $f(x) = x^2$ is $2x$, so the **integral** $\int g(x)dx$ of $g(x) = 2x$ should be x^2.
But: We also learnt that the differential of $\tilde{f}(x) = x^2 + 9$ is $2x$, so the **integral** $\int g(x)dx$ of $g(x) = 2x$ should be $x^2 + 9$?!
Thus, in the same way as we *lose* constants in the process of differentiating, we have to *replace* the constant when integrating.
This "unknown" constant is symbolized by the letter c.
We write:

$$\int g(x)dx = \int 2xdx = x^2 + c$$

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What is the definite integral?

If we denote by $F(x)$ the indefinite integral $\int f(x)dx$ of some function f, the **definite integral** of f between the values $x = a$ and $x = b$ is:

$$\int_a^b f(x)dx = F(b) - F(a)$$

Example

- Since $\int 2xdx = x^2 + c$, we have $\int_a^b 2xdx = b^2 + c - (a^2 + c) = b^2 - a^2$.
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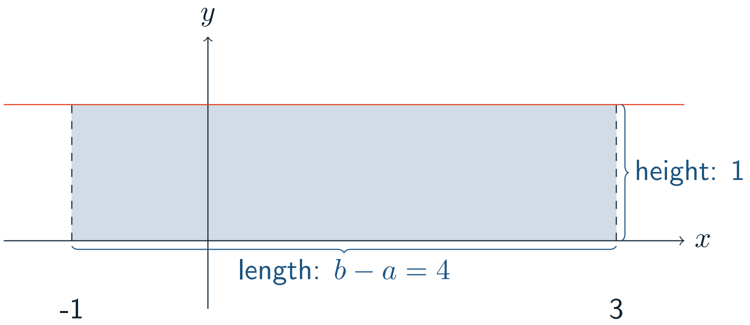
What does it represent?

Let's have a look at $\int_a^b 1dx = b - a$ again, for $b = 3$ and $a = -1$:



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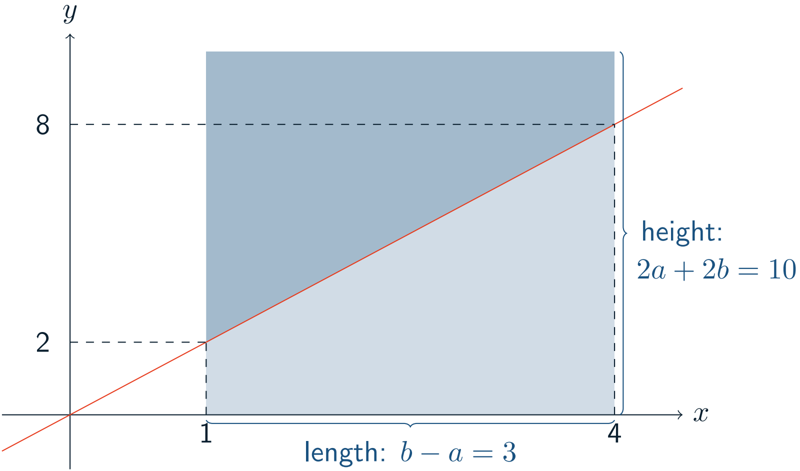
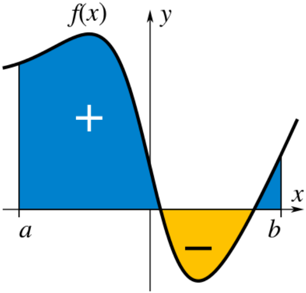
Let's have a look at $\int_a^b 2x dx = b^2 - a^2$ again, for $b = 4$ and $a = 1$:



Definite integrals and areas

For a function f that always stays above the x-axis, the definite integral $\int_a^b f(x)dx$ corresponds to the area between the curve corresponding to $f(x)$, the lines $x = a$, $x = b$ and the x-axis.

However, if f goes below the x-axis at some points, the integral counts a "weighted area", where every part under the x-axis gets a negative weight.



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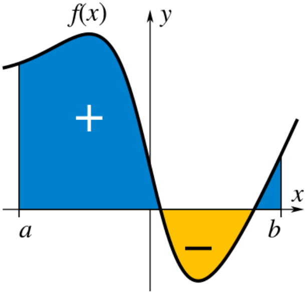
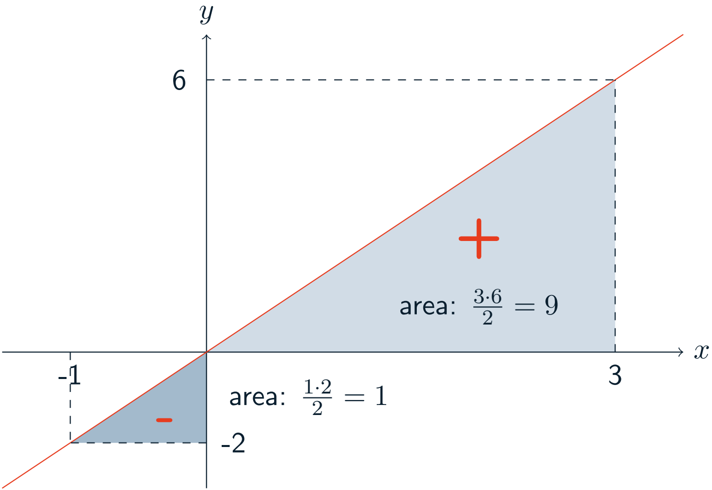


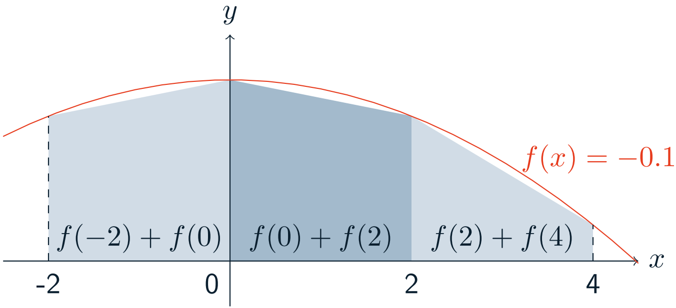
Example

Now let's evaluate $\int_a^b 2x dx = b^2 - a^2$ for $b = 3$ and $a = -1$:



Approximation with trapeziums

The area under a curve or, equivalently, a definite integral can be approximated using the **trapezium rule**:

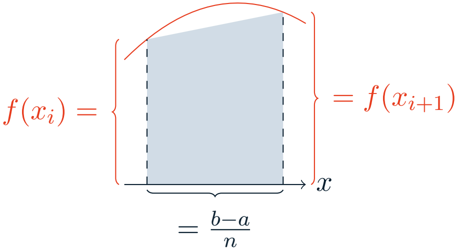


The trapezium rule with n parts

The general trapezium rule with n parts is given as follows:

$$\int_a^b f(x)dx \approx \sum_{i=0}^{n-1} \frac{1}{2}\frac{b-a}{n} \cdot \left(f\left(a + \frac{i\cdot(b-a)}{n}\right) + f\left(a + \frac{(i+1)\cdot(b-a)}{n}\right)\right)$$
$$\approx \frac{b-a}{2n}(f(x_1) + 2f(x_2) + \ldots + 2f(x_n) + f(x_{n+1})),$$

where the x_i are spaced out evenly on the interval $[a,b]$ and $a = x_1$ and $b = x_{n+1}$.

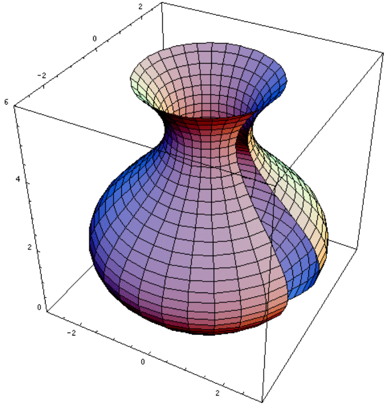


Indefinite integrals of some simple functions

- $f(x) = k \qquad \int f(x)dx = kx + c$,
  where $k \in \mathbb{R}$ is some constant
- $f(x) = x^k \qquad \int f(x)dx = \frac{x^{k+1}}{k+1} + c$,
  where $k \in \mathbb{Z}$, $k \neq 1$
- $f(x) = \frac{1}{2\sqrt{x}} \qquad \int f(x)dx = \sqrt{x} + c$
- $f(x) = \frac{1}{x} \qquad \int f(x)dx = \ln(x) + c$
- $f(x) = e^x \qquad \int f(x)dx = e^x + c$
- $f(x) = \cos(x) \qquad \int f(x)dx = \sin(x) + c$
- $f(x) = \sin(x) \qquad \int f(x)dx = -\cos(x) + c$

Solid figures

We can use integration not only to calculate areas but also to calculate volumes of certain solids, namely **solids of revolution**. These are solid figures obtained by rotating a curve around an axis.



▸ Link

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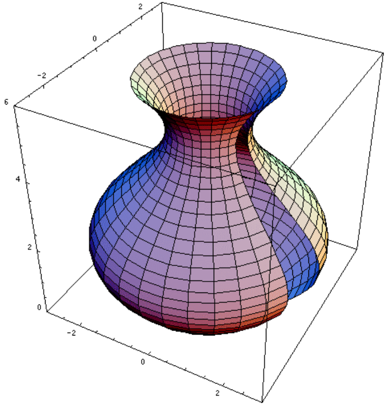


Link

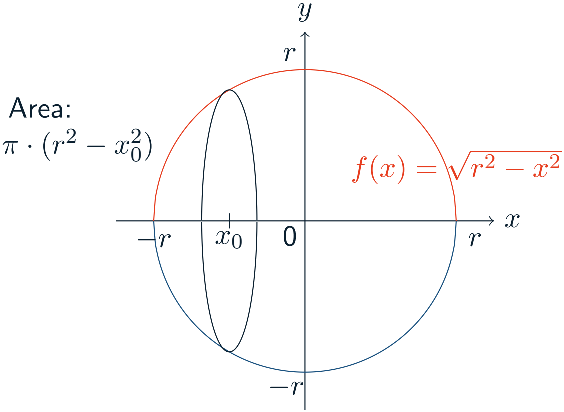
Idea

We cut up the solid of revolution into slices that are perpendicular to the rotation axis; these will be **disks** of radius $f(x)$. Then we calculate the area of these disks and "sum up" the areas for all these infinitely many disks, this is done by integration.
This leads to the following formula for the volume of a solid of revolution that is obtained by rotating the curve of the function $f(x)$ around the x-axis for values of x between a and b:

$$V = \pi \cdot \int_a^b (f(x))^2 dx.$$

Example: Volume of a sphere

We want to use this method to rediscover the formula for the volume of a sphere with radius r.



Example: Volume of a sphere

Plugging this into the formula, we obtain:

$$
\begin{aligned}
V &= \pi \cdot \int_{-r}^{r} r^2 - x^2 dx = \pi \cdot \int_{-r}^{r} r^2 dx - \pi \cdot \int_{-r}^{r} x^2 dx \\
&= \pi \cdot r^2 \left[x\right]_{-r}^{r} - \pi \left[\frac{x^3}{3}\right]_{-r}^{r} \\
&= \pi r^2 (r - (-r)) - \pi \cdot \left(\frac{r^3}{3} - \frac{-r^3}{3}\right) \\
&= 2\pi r^3 - \pi r^3 \frac{2}{3}
\end{aligned}
$$

$$V = \frac{4}{3}\pi r^3.$$

Outline

What is a vector?

From a mathematical point of view, a **vector** of dimension n (a natural number) is simply an ordered list of n items, most often from $\mathbb{R}$, but possibly also from another set.
We use the following notation:

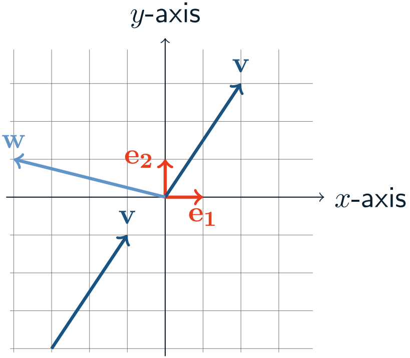
$$\mathbf{v} = \vec{v} = \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \begin{pmatrix} a & b & c \end{pmatrix}^T = (a, b, c)^T,$$

this is a vector of dimension 3.
We can define a multiplication of numbers – scalars – with vectors and an addition of two vectors.

What does a vector represent?

Vectors can be represented as points in n-dimensional space. For instance, for $n = 2$:

$$\mathbf{e_1} = (1,0)^T$$
$$\mathbf{e_2} = (0,1)^T$$
$$\mathbf{v} = (2,3)^T$$
$$\mathbf{w} = (-4,1)^T$$



A vector is characterized by its **length** or **magnitude** and its **direction**. If two vectors have the same length and magnitude, they are the same.

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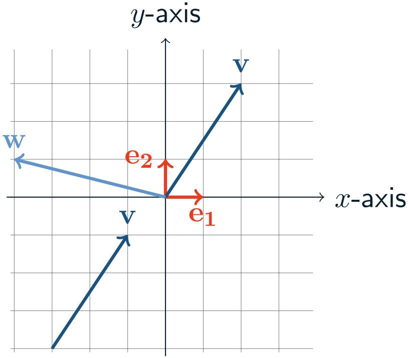
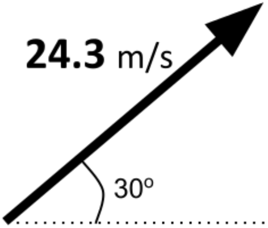


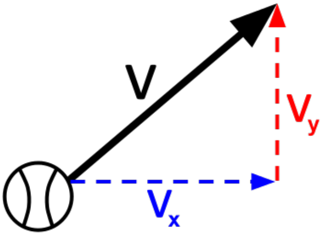
A vector is characterized by its **length** or **magnitude** and its **direction**. If two vectors have the same length and magnitude, they are the same.

Vectors can represent movement in space

Speed: distance covered per unit of time $v = d/t$
For example 5 m/s, 30 km/h
This is a scalar quantity

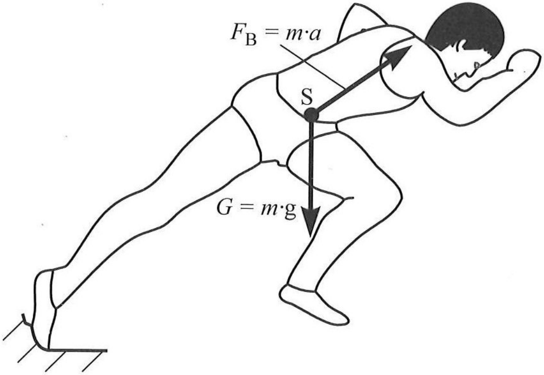
Velocity: speed + direction of motion
For example: 30 km/h South
This can be represented by a vector





Vectors as forces

Vectors can be used to represent forces, since vectors have both magnitude and direction. This is important in the field of **biomechanics**. The unit of measurement for forces is newton (N).

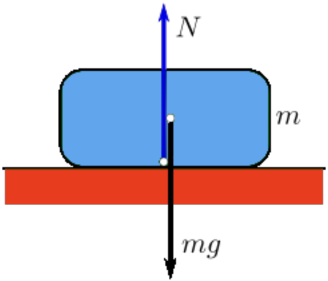


Some important forces

- The **weight** of an object is the force on the object due to **gravity**. Its magnitude is the product of the mass m of the object and the magnitude of the gravitational acceleration $g \approx 9.81 m/s^2$;thus: $G = mg$. It acts on the centre of gravity of the object and is pointed towards the Earth's center.
- The **acceleration** acts on the centre of gravity of the object and its direction corresponds to the direction of the acceleration. Its magnitude is the product of the mass m of the object and the magnitude of the acceleration a;thus: $F_a = ma$.

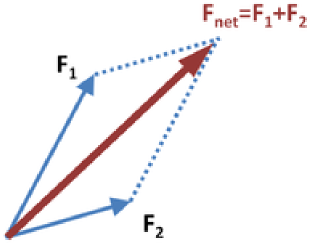
Two laws of mechanics (1)

- The law of **reaction**, also known as Actio est Reactio.
  If one object exerts a force on another object, then the second object exerts an equal and opposite reaction force on the first.



Two laws of mechanics (2)

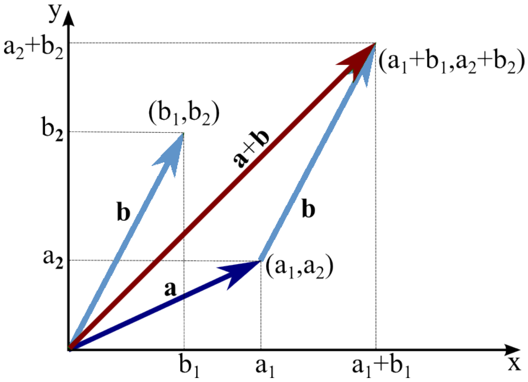
- When two forces act on the same point they can be replaced by a single force. This resulting force then causes the same result as the two initial forces. The **parallelogram of force**, tells us how to determine the resulting force:



Vector addition in general

Given two vectors $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ of the same dimension, their sum is calculated as follows:

$$\mathbf{z} = \mathbf{v} + \mathbf{w} = (v_1 + w_1, v_2 + w_2, \ldots, v_n + w_n)$$



Scalar multiplication

A **scalar** is a real number. Vectors can be multiplied with scalars in the following way:

$$k \cdot \mathbf{v} = k \cdot (v_1, v_2, \ldots, v_n) = (k \cdot v_1, k \cdot v_2, \ldots, k \cdot v_n)$$

Example

If $\mathbf{v} = (2,3)^T$ then $2 \cdot \mathbf{v} = (4,6)^T$ and $-\mathbf{v}/3 = (-2/3, -1)^T$.

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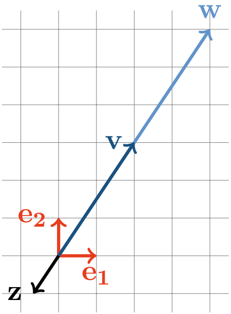
Vectors with the same direction

Two vectors $\mathbf{v}$ and $\mathbf{w}$, have the same direction if there is some **positive scalar** $k > 0$ such that:

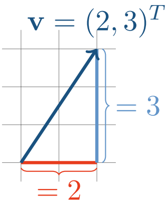
$$\mathbf{v} = k \cdot \mathbf{w}.$$

$$\begin{aligned} \mathbf{v} &= (2,3)^T \\ \mathbf{w} &= (4,6)^T = 2 \cdot \mathbf{v} \\ \mathbf{z} &= (-2/3,-1)^T = -\mathbf{v}/3 \end{aligned}$$



$\mathbf{v}$ and $\mathbf{w}$ have the same direction, $\mathbf{v}$ and $\mathbf{z}$ have opposite direction

Calculating the length of a vector



Pythagoras' theorem:

$$||\mathbf{v}||^2 = 2^2 + 3^2 = 13$$
$$||\mathbf{v}|| = \sqrt{13}$$

The general formula

Given a vector $\mathbf{v} = (v_1, v_2, \dots, v_n)^T$, its length is given as follows:

$$||\mathbf{v}|| = \sqrt{v_1^2 + v_2^2 + \ldots + v_n^2}.$$

Unit vectors

Definiton

A **unit vector** is a vector with length 1.

Given a vector $\mathbf{v}$ one can obtain a unit vector with same direction as follows:

$$\hat{\mathbf{v}} = \frac{\mathbf{v}}{||\mathbf{v}||}$$

This vector is called the **normalized vector** of $\mathbf{v}$. One often uses the following special unit vectors in 2-dimensional space:

$$\mathbf{e_1} = (1,0)^T \text{ and } \mathbf{e_2} = (0,1)^T$$

and in 3-dimensional space:

$$\mathbf{e_1} = (1,0,0)^T, \quad \mathbf{e_2} = (0,1,0)^T \text{ and } \mathbf{e_3} = (0,0,1)^T$$

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Scalar product

The **scalar product** (also: **dot product**) associates to two vectors $\mathbf{v} = (v_1, v_2, \ldots, v_n)$ and $\mathbf{w} = (w_1, w_2, \ldots, w_n)$ a scalar in the following way:

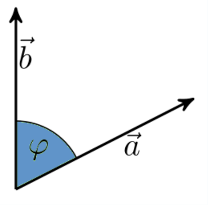
$$\langle \mathbf{v}, \mathbf{w} \rangle = \mathbf{v} \cdot \mathbf{w} = v_1 \cdot w_1 + v_2 \cdot w_2 + \ldots + v_n \cdot w_n.$$

If $\mathbf{v} \neq \mathbf{0}$, $\mathbf{w} \neq \mathbf{0}$ and $\mathbf{v} \cdot \mathbf{w} = 0$, this means that $\mathbf{v}$ and $\mathbf{w}$ are **orthogonal** (perpendicular).

Scalar product and angles

If φ is the angle between the two vectors $\mathbf{a}$ and $\mathbf{b}$, the scalar product can also be defined as follows:

$$\mathbf{a} \cdot \mathbf{b} = \cos(\varphi) \cdot ||\mathbf{a}|| \cdot ||\mathbf{b}||.$$



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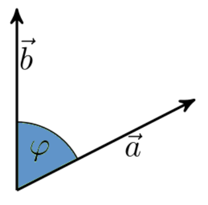

Scalar product - Example

- $(1,0)^T \cdot (0,1)^T) = 1 \cdot 0 + 0 \cdot 1 = 0$
  These two vectors are orthogonal.
- $(x,y)^T \cdot (-y,x)^T = -xy + yx = 0$
  These two vectors are also orthogonal.
- $(1,2,-2)^T \cdot (3,0,4)^T = 1 \cdot 3 + 2 \cdot 0 - 2 \cdot 4 = 3 - 8 = -5.$
  These two vectors are not orthogonal.
  What is the angle between these two vectors?
  $||(1,2,-2)^T|| = \sqrt{1+4+4} = \sqrt{9} = 3$ and
  $||(3,0,4)^T|| = \sqrt{9+0+16} = \sqrt{25} = 5$ Thus
  $\cos(\varphi) = -5/(5 \cdot 3) = -1/3$ and $\varphi \approx 1.91$ rad $\approx 109.4°$.

Cross product

The **cross product** of two vectors $\mathbf{v} = (v_1, v_2, v_3)$ and $\mathbf{w} = (w_1, w_2, w_3)$ returns a vector that is orthogonal to the other two.

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} \times \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} v_2 w_3 - v_3 w_2 \\ v_3 w_1 - v_1 w_3 \\ v_1 w_2 - v_2 w_1 \end{pmatrix}$$

Cross product - Example

- The cross product of the two vectors $\mathbf{e_1}$ and $\mathbf{e_2}$ gives $\mathbf{e_3}$:

$$\begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix} \times \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \cdot 0 - 0 \cdot 1 \\ 0 \cdot 0 - 1 \cdot 0 \\ 1 \cdot 1 - 0 \cdot 0 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}$$

- If the vectors have the same direction or one has zero length, then their cross product is zero:

$$\begin{pmatrix} 2 \\ 4 \\ -2 \end{pmatrix} \times \begin{pmatrix} -1 \\ -2 \\ 1 \end{pmatrix} = \begin{pmatrix} 4 \cdot 1 - (-2) \cdot (-2) \\ (-2) \cdot (-1) - 2 \cdot 1 \\ 2 \cdot (-2) - 4 \cdot (-1) \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}$$

Definition

An $n \times m$ or (n, m)-matrix is a table or **array** consisting of n rows and m columns where every field contains a real number. We use the following notation

$$A = (a_{ij})_{1 \leq i,j \leq 3} = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix},$$

this is a $(3, 3)$ matrix. For the entry in the i-th row and the j-th column we write a_{ij}.
We will see later on what matrices can be used for.

The transpose of a matrix

To a matrix A we associate its **transpose** matrix A^T by mirroring the elements along the diagonal from the top left to the bottom right corner. This means that the rows in A are the columns in A^T and vice-versa.

$$\text{With } A = \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & i \end{pmatrix} \text{ we have } A^T = \begin{pmatrix} a & d & g \\ b & e & h \\ c & f & i \end{pmatrix}$$

If A is of size $n \times m$, then A^T is of size $m \times n$.

Example

$$\text{With } B = \begin{pmatrix} 1 & 0 & -1 \\ 2 & 1 & 0 \end{pmatrix} \text{ we have } B^T = \begin{pmatrix} 1 & 2 \\ 0 & 1 \\ -1 & 0 \end{pmatrix}$$

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Special kinds of matrices (1)

- If $m = 1$, we have a **column vector** as in the previous section.
- If $n = 1$, we have a **row vector**.
- If $m = n$, we have a **square** matrix.
- A square matrix A is called **symmetric**, if $A^T = A$.

Example

$$A = \begin{pmatrix} 1 & 2 \\ 2 & 1 \end{pmatrix} \text{ or } B = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 0 & 4 \\ 3 & 4 & -1 \end{pmatrix}$$

Special kinds of matrices (2)

Among the symmetric matrices we distinguish the following types:

- Matrices where all elements *above* the diagonal are zero are called **lower triangular** matrices and matrices where all elements *below* the diagonal are zero are called **upper triangular** matrices.

$$L = \begin{pmatrix} 2 & 0 & 0 \\ -1 & 0 & 0 \\ 7 & 0 & 3 \end{pmatrix} \text{ or } U = \begin{pmatrix} -2 & 8 & 0 \\ 0 & 3 & 1 \\ 0 & 0 & -1 \end{pmatrix}$$

- **Diagonal** matrices - these are matrices where all elements except those lying on the diagonal are zero. These matrices are both upper and lower triangular matrices. For example:

$$A = \begin{pmatrix} 2 & 0 \\ 0 & 1 \end{pmatrix} \text{ or } B = \begin{pmatrix} -2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

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Special kinds of matrices (3)

The diagonal matrix of size n where all diagonal elements are equal to 1 is the **identity** matrix I_n.

$$I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ and } I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

The columns (and equivalently the rows) of I_n are the vectors $\mathbf{e_1}$, $\mathbf{e_2}$, ..., $\mathbf{e_n}$ of dimension n.

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Addition of matrices

Two matrices A and B of the same size can be added up by adding up their elements. This means that the element c_{ij} at the i-th row and j-th column of $C = A + B$ is equal to $a_{ij} + b_{ij}$.

Example

$$A+B = \begin{pmatrix} 1 & 4 & -1 \\ 0 & -1 & 0 \\ 5 & 0 & 3 \end{pmatrix} + \begin{pmatrix} -1 & 0 & 1 \\ 2 & -2 & 4 \\ 5 & -2 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 4 & 0 \\ 2 & -3 & 4 \\ 10 & -2 & 3 \end{pmatrix}$$

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Scalar multiplication

In the same way as we can multiply vectors with scalars, we can do so with matrices. Multiplying a matrix A by some scalar k, means that every element of A is multiplied by k.

Example

$$2 \cdot \begin{pmatrix} 1 & 4 & -1 \\ 0 & -1 & 0 \\ 5 & 0 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 8 & -2 \\ 0 & -2 & 0 \\ 10 & 0 & 6 \end{pmatrix}$$

$$-\frac{1}{4} \cdot \begin{pmatrix} 1 & 4 & -1 \\ 0 & -1 & 0 \\ 5 & 0 & 3 \end{pmatrix} = \begin{pmatrix} -1/4 & -1 & 1/4 \\ 0 & 1/4 & 0 \\ -5/4 & 0 & -3/4 \end{pmatrix}$$

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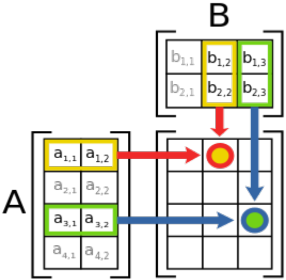
Multiplication of matrices

Two matrices A and B can also be multiplied by each other if they have matching dimensions: if A has dimension (n, m), B has to have dimension (m, k). Their product $C = A \cdot B$ will then have dimension (n, k).

The element c_{ij} in the i-th row and j-th column of C is obtained by calculating the *scalar product* of the vector in the i-th row in A with the vector in the j-th column of B.



Multiplication of matrices - Example

$$
\begin{aligned}
A \cdot B &= \begin{pmatrix} 1 & 4 & -1 \\ 0 & -1 & 0 \\ 5 & 0 & 3 \end{pmatrix} \cdot \begin{pmatrix} -1 & 0 \\ 2 & -2 \\ 5 & -2 \end{pmatrix} \\
&= \begin{pmatrix} 1 \cdot (-1) + 4 \cdot 2 + (-1) \cdot 5 & 1 \cdot 0 + 4 \cdot (-2) + (-1) \cdot (-2) \\ 0 \cdot (-1) + (-1) \cdot 2 + 0 \cdot 5 & 0 \cdot 0 + (-1) \cdot (-2) + 0 \cdot (-2) \\ 5 \cdot (-1) + 0 \cdot 2 + 3 \cdot 5 & 5 \cdot 0 + 0 \cdot (-2) + 3 \cdot (-2) \end{pmatrix} \\
&= \begin{pmatrix} 2 & -6 \\ -2 & 2 \\ 10 & -6 \end{pmatrix}
\end{aligned}
$$

Online tool

Matrices as linear functions

Given a matrix A of dimension (n, m), it defines a **linear function** on the set of all m-dimensional vectors in the following way:

$$A(\mathbf{x}) = A \cdot \mathbf{x} = \mathbf{y}.$$

The output vector $\mathbf{y}$ is then of dimension n.
The elements of A have the following meaning: the j-th column in A is the image $A(\mathrm{e_j})$ of the vector $\mathrm{e_j}$.

Example

$$\begin{pmatrix} -1 & 0 \\ 2 & -2 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 0 \end{pmatrix} = \begin{pmatrix} -1 \\ 2 \end{pmatrix}, \begin{pmatrix} -1 & 0 \\ 2 & -2 \end{pmatrix} \cdot \begin{pmatrix} 0 \\ 1 \end{pmatrix} = \begin{pmatrix} 0 \\ -2 \end{pmatrix}$$

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Examples

The following linear functions on vectors are essential for image processing:

- **Rotating:** The following matrix describes a rotation by the angle φ about the origin in counter-clockwise direction (in the 2-dimensional plane):

$$\begin{pmatrix} \cos(\varphi) & -\sin(\varphi) \\ \sin(\varphi) & \cos(\varphi) \end{pmatrix}$$

- **Scaling:** The following matrix describes a scaling by the factor α (both in $x-$ and in y-direction in the 2-dimensional plane):

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