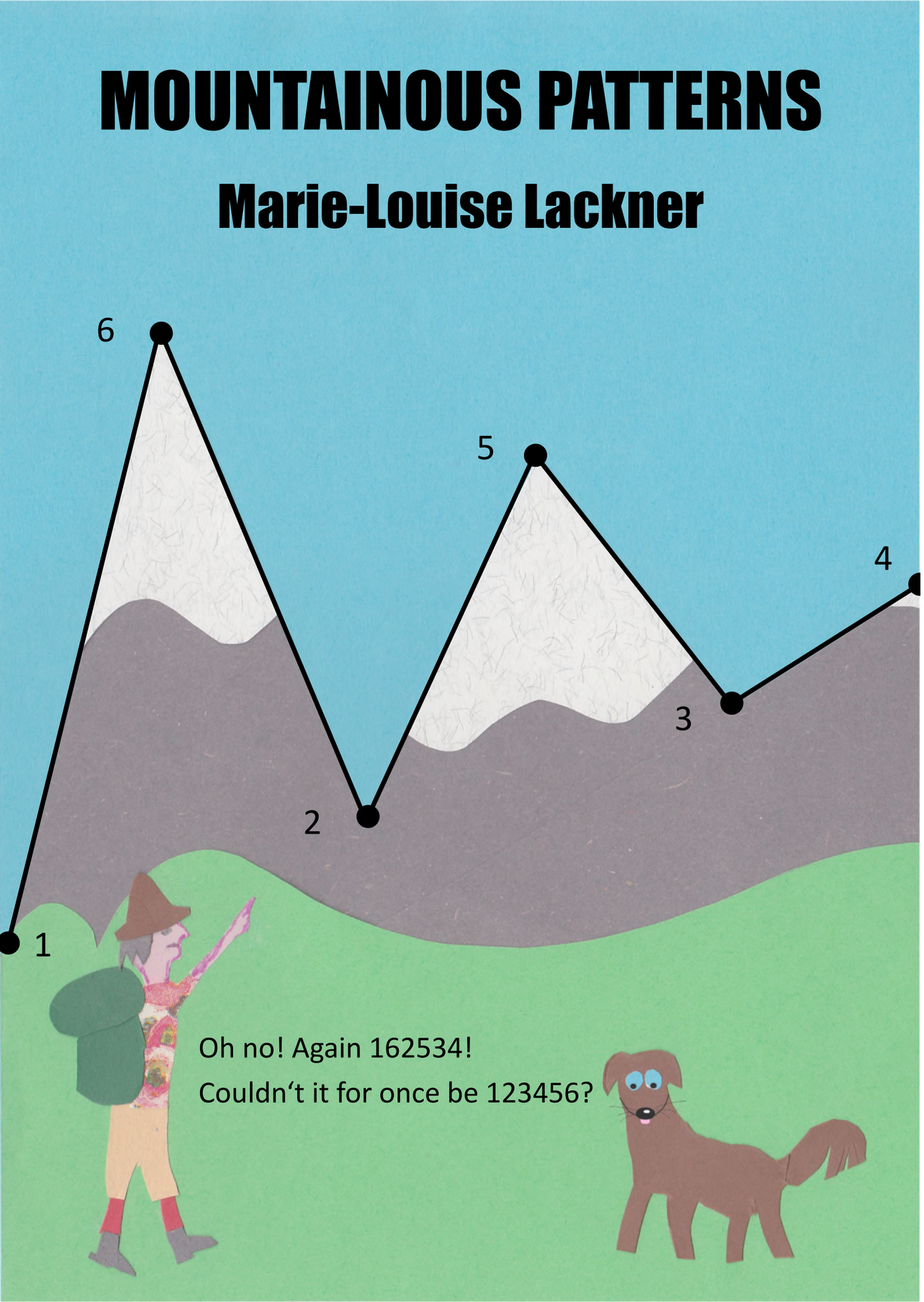

MOUNTAINOUS PATTERNS

A playful introduction to
pattern avoidance in permutations

by

Marie-Louise Lackner

(née Bruner)

INTRODUCTION

Introduction p.11

These riddles are an attempt to make the topic of my master‘s thesis entitled "Restricted permutations on multisets" accessible to a broad public that does not necessarily have a mathematical background. MOUNTAINOUS PATTERNS explains what pattern avoidance is about and offer an illustration of some results presented in my thesis. I very much hope that these riddles give newcomers a playful insight into the fascinating topic of pattern avoidance.

The riddles in this booklet are grouped into two parts. In the first part, pawns have to be placed in a square grid following certain rules. This corresponds to Chapters 2 and 3 in my master‘s thesis and illustrates pattern avoidance in (ordinary) permutations. In the second part, the square grids are replaced by rectangular ones. This corresponds to Chapters 2, 4 and 5 in my thesis where ordinary permutations have been replaced by permutations on regular multisets.

Vienna, May 2011

INTRODUCTION

The times of my master’s studies at TU Wien seem far away now; in the meantime I obtained a PhD degree in mathematics and have worked as a postdoctoral researcher in combinatorics for one year. The interest for permutation patterns is, however, still as fresh as when I first got it contact with this topic. A substantial part of my research so far was devoted to enumerative and computational aspects of permutation patterns and I am fascinated by the wealth of differently flavoured results that this field has produced in recent years.

I had printed roughly 30 copies of the first edition of MOUNTAINOUS PATTERNS but these were soon sold or given away as presents and I was left with a single sample copy, full of comments and corrections, that is nearly falling apart now... Boosted by several positive reactions of friends and colleagues who had found the electronic version on my webpage, I decided to pick up MOUNTAINOUS PATTERNS again. I slightly revised the first edition, got rid of a couple of small mistakes (there are probably still some hiding! - do let me know if you find any) and decided to actually publish this booklet.

Oxford, June 2016

THANKS

My special thanks go to my father-in-law Hans Lackner who, without knowing it, inspired me to think about pattern avoidance problems as mathematical riddles and who gave the impulse to MOUNTAINOUS PATTERNS. I also want to thank my father Gerhard Bruner and my husband Martin Lackner for their creative input, encouragement and feedback.

Alois Panholzer was the one who introduced me to the fascinating topic of Pattern avoidance during his Combinatorics seminar at TU Wien in the spring of 2010. My thanks go to him for being an inspirational teacher and for strongly supporting my research in this field during my Master's and PhD studies.

I highly appreciate the positive reactions that I got from many friends and colleagues whom I cannot all mention by name here. Without you the publication of this booklet would never have happened.

Last but not least, I wish to acknowledge the financial support from the Austrian Science Fund FWF who funded my research both during my master‘s and doctoral studies (project numbers P25337-N23 and S9608).

CONTENTS

HOW TO USE THIS BOOKLET

It is not strictly necessary to follow the order of the riddles in this booklet, you can start wherever you like. However, the difficulty of the riddles increases as you move trough the book, so you might prefer to start at the very beginning. Moreover, parts 1 and 2 are basically independent from each other, but you will probably find the second part more challenging. So that you can get a feeling for the difficulty of individual riddles, I used the following specification with stars:

☆☆☆☆☆	EASY-PEASY
★☆☆☆☆	WARM-UP
★★☆☆☆	HALF-WAY
★★★☆☆	HEAD-SCRATCHER
★★★★☆	TOUGH NUT
★★★★★	BRAIN-KILLER

Also, the names of riddles are an indication for how hard they are. The names all refer to places in Austria where you can go for a pleasant stroll, an enjoyable walk in the woods or a challenging hike in the mountains. For instance, Rathauspark (p. 14) is a park in the city centre of Vienna, right next to the city hall. This is a place where I spent a lot of time as a young child and even toddlers can run around there without any trouble. On the other hand, Piz Buin (p. 54) is the highest mountain in Vorarlberg, the westernmost province of Austria and reaching its peak requires a good deal of experience, strength and endurance.

In case you need a hint or want to read up on the theoretical background of MOUNTAINOUS PATTERNS, the chapters, definitions, theorems or pages of my master‘s thesis can be helpful. See the red boxes in the top right corner of every page for reference. Moreover, a list of references can be found on page 70. Please do contact me if you have any questions or remarks.

I hope you will enjoy solving these riddles and get a playful insight into the exciting field of pattern avoidance.

GOOD LUCK!

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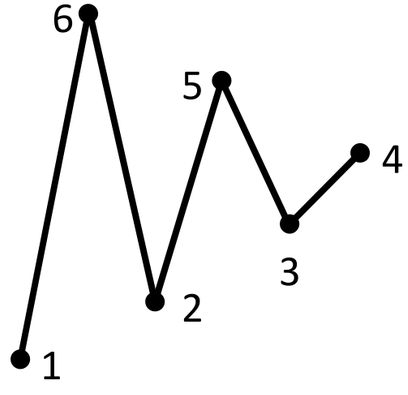
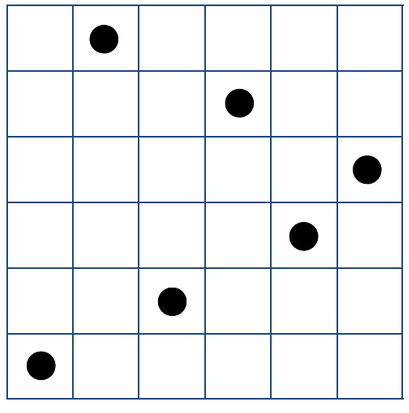
BACKGROUND

Preliminaries p.15

For those who are interested in some mathematical background: The objects analysed within pattern avoidance are so-called "restricted permutations". Permutations are the objects at the heart of discrete mathematics and algebra that can either be seen as linear orderings or as bijective maps from a set onto itself. Restricted permutations are permutations with a special inherent structure, namely such a structure that does not allow for certain patterns to appear. Permutations can be represented in several different ways - as sequences, plots, graphs... On the cover, the permutation 162534 is represented with the help of its graph as a function. For a better visualization, we will use grids in which pawns have been placed throughout this booklet. The permutation can be read off easily: For instance, the pawn in the third column (from left) lies at „height“ two (the second row from below), so the third element of the permutation is 2. The two pictures below thus represent the same permutation on six elements.





Chapters 2 & 3

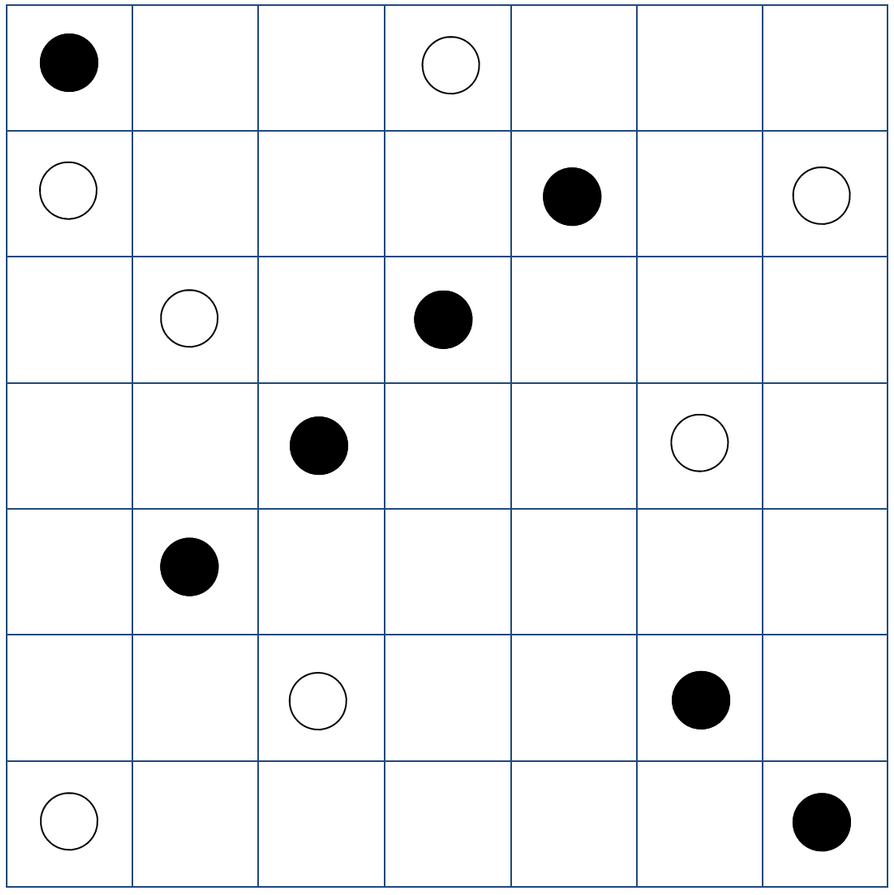
Part 1
SQUARE GRIDS

(Permutations on ordinary sets)

RULES

Preliminaries p.15

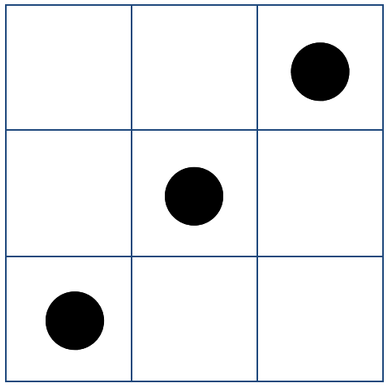
We are given a square grid such as the one presented below and as many pawns as there are rows/columns in the grid. In our example we thus have seven pawns. These pawns have to be placed in the empty spaces following the rule: in every column and in every row there has to be exactly one pawn. For instance, placing the pawns like the black ones is allowed but placing them like the white ones is forbidden.

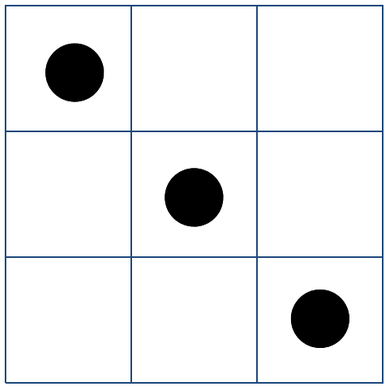


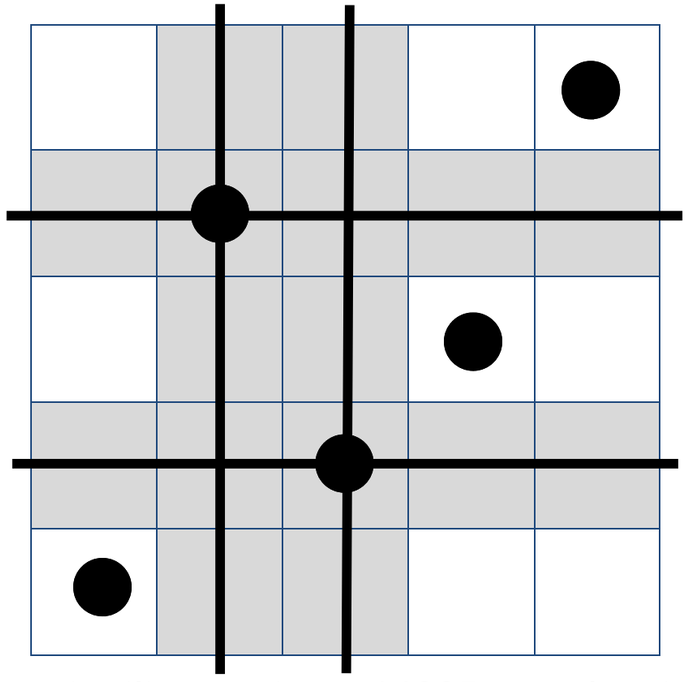
For those who like numbers: the black configuration corresponds to the permutation 7345621.

Chapter 3 p. 33

We say that a pattern corresponding to a smaller grid like the 3x3-grids below is contained in a larger grid if, by deleting some rows and some columns, we obtain a grid in the size of the pattern that looks exactly like the pattern-grid.





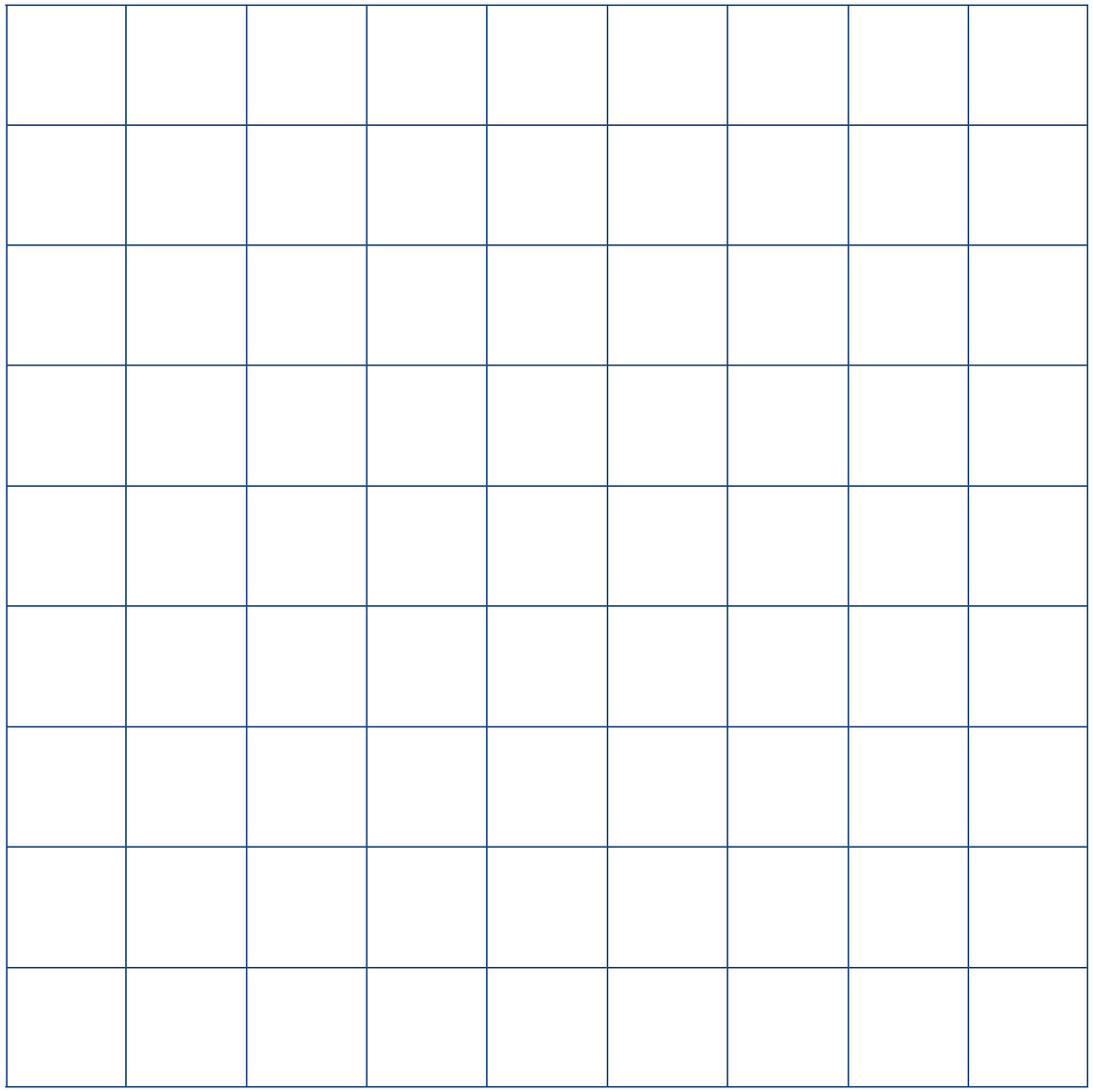


In this example , we can see that the 5x5-grid contains the pattern on the left by deleting the rows and columns marked in grey. This is not possible for the pattern on the right. Therefore we say that the pattern on the right is avoided. For those who like numbers: 14235 contains the pattern 123 but avoids the pattern 321.

STADTPARK

Preliminaries p.15

Forbidden pattern: none

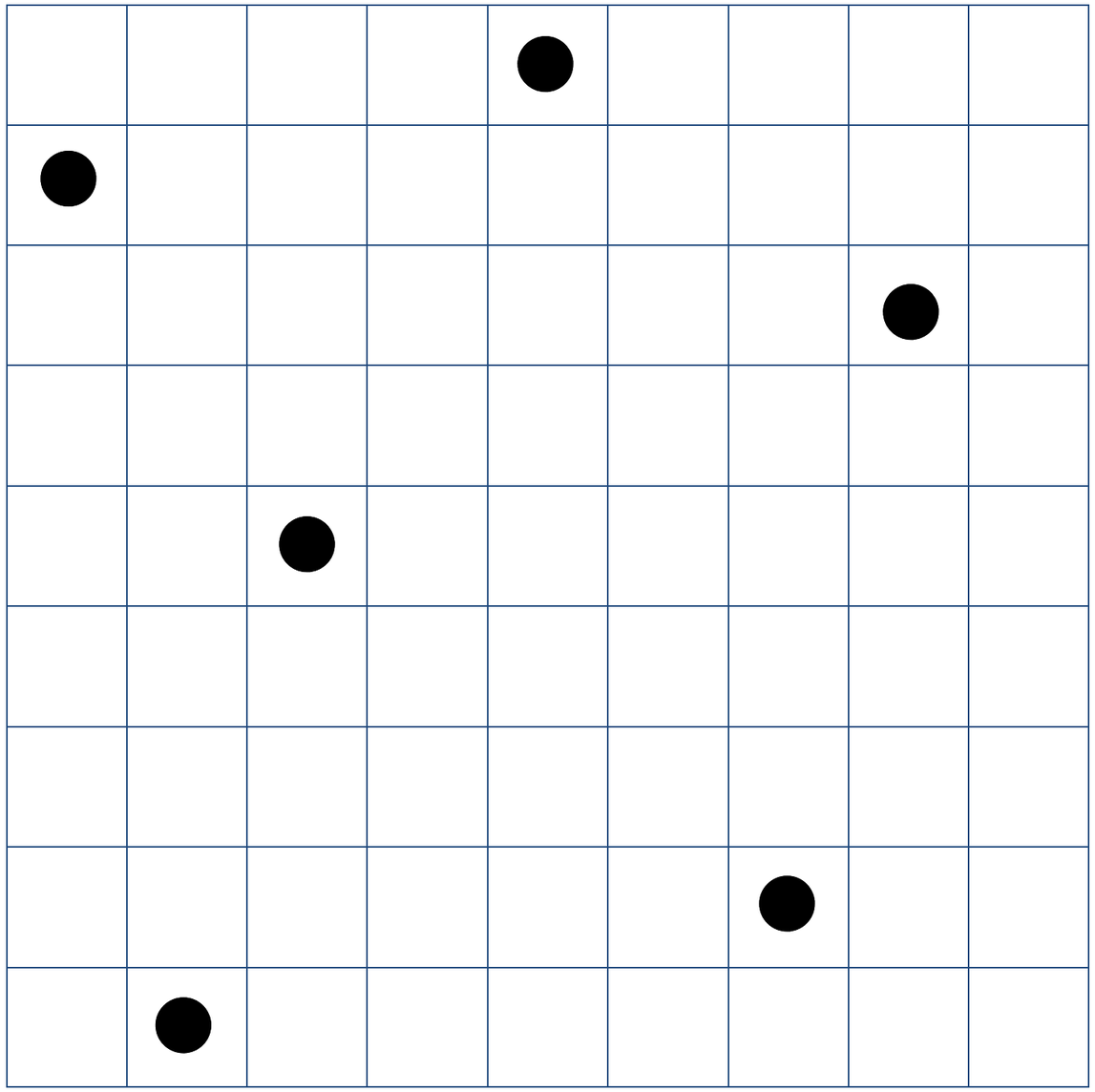


☆☆☆☆☆ Fill in nine pawns in an allowed way.

VOLKSGARTEN

Preliminaries p.15

Forbidden pattern: none



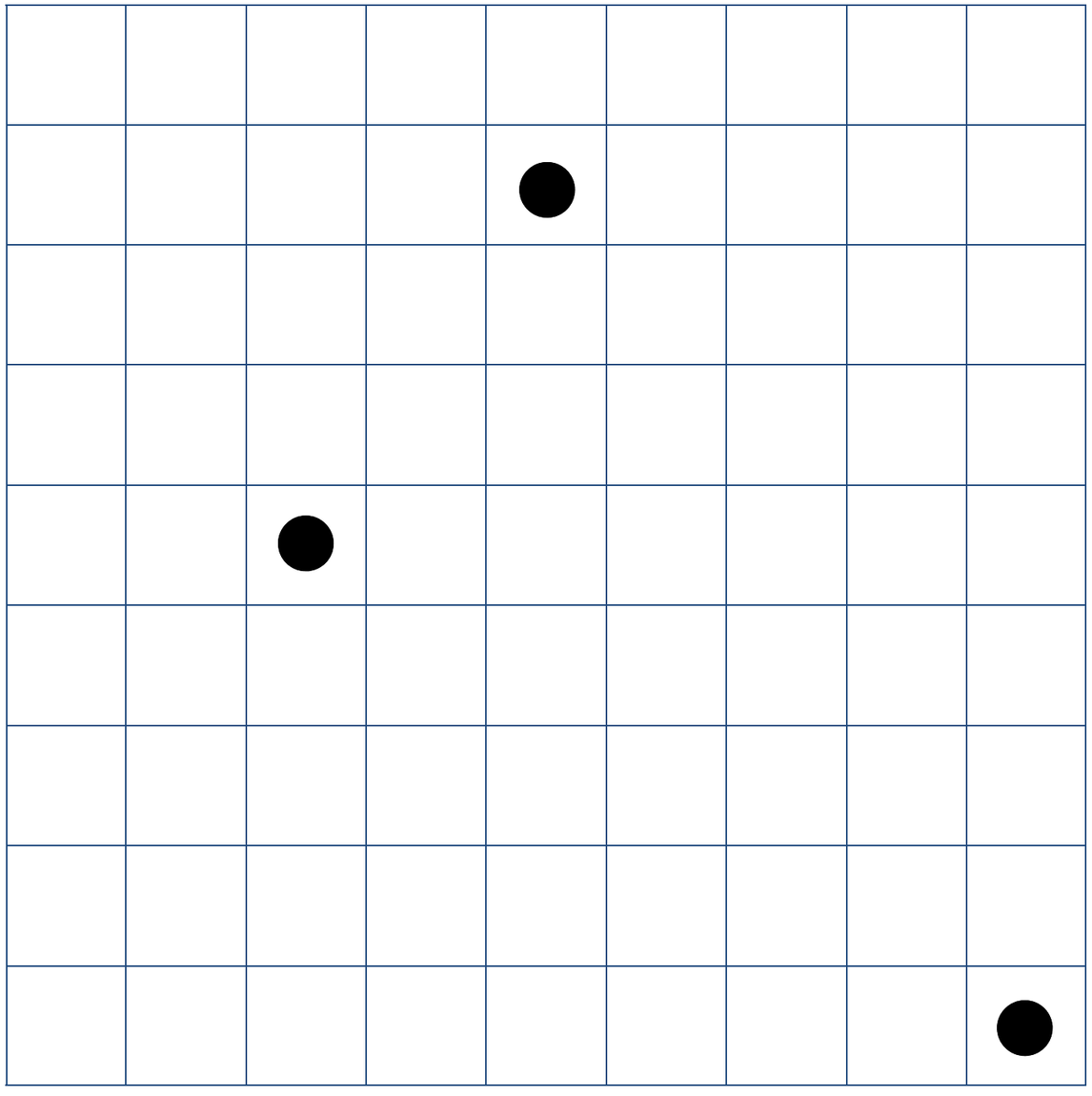
☆☆☆☆☆ Fill in the remaining pawns in an allowed way.

★☆☆☆☆ In how many different ways can this be done?

RATHAUSPARK

Preliminaries p.15

Forbidden pattern: none



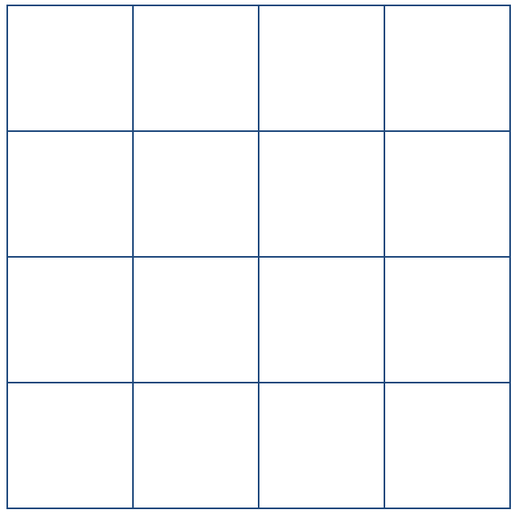
☆☆☆☆☆ Fill in the remaining pawns in an allowed way.

★★☆☆☆ In how many different ways can this be done?

PRATER

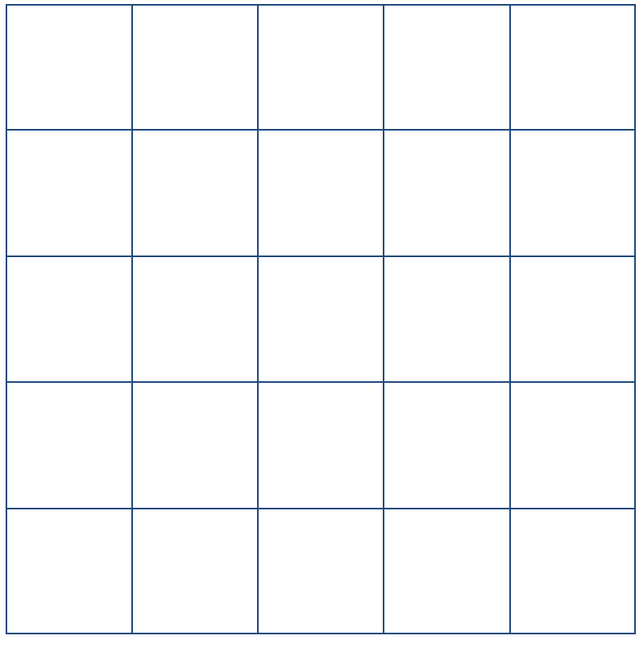
Preliminaries p.15

Forbidden pattern: none



★☆☆☆☆

In how many different allowed ways may four pawns be placed in this grid?



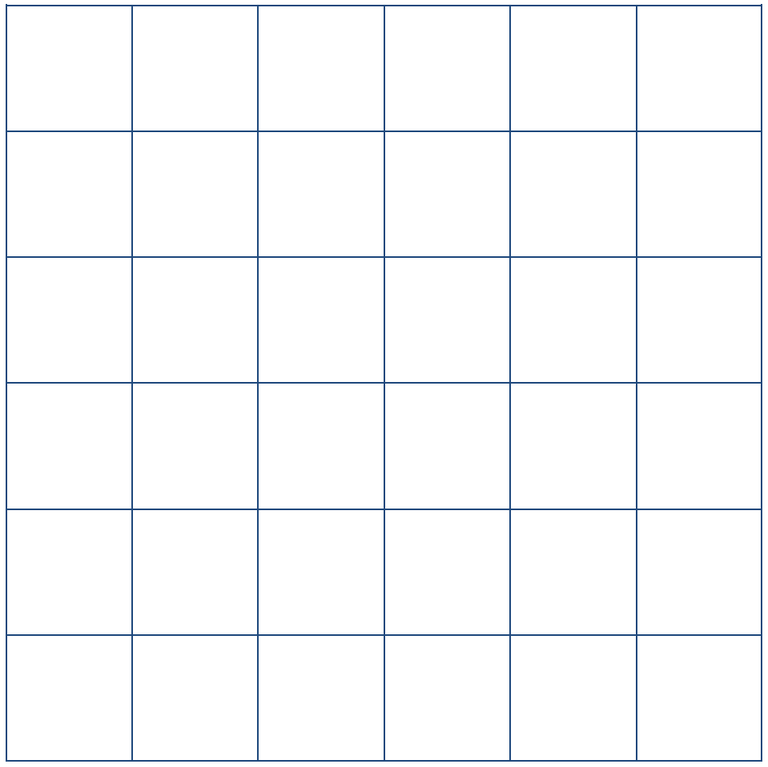
★☆☆☆☆

In how many different ways can this be done here?

SCHÖNBRUNN

Preliminaries p. 15

Forbidden pattern: none



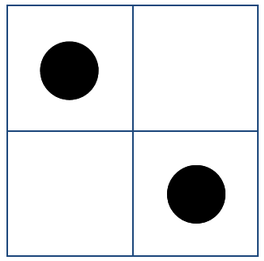
★☆☆☆☆ In how many different allowed ways may six pawns be placed in this grid?

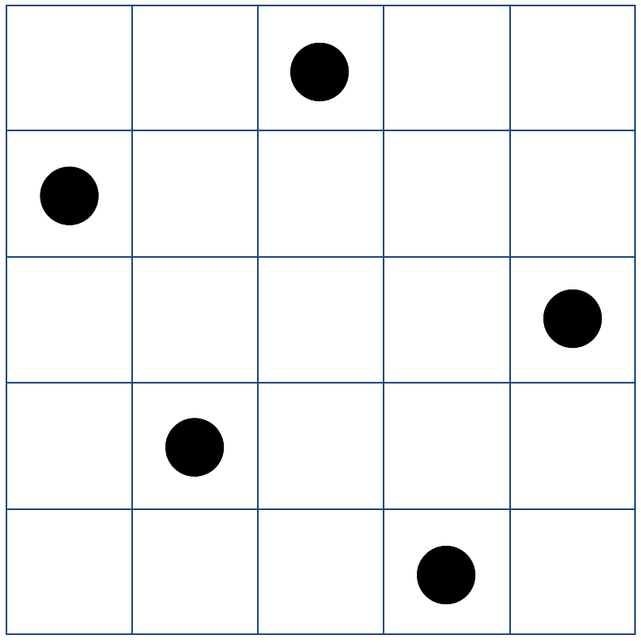
★★☆☆☆ If the size of this grid was not 6x6 but 15x15 how many different possibilities of placing 15 pawns would we then have ?

SOPHIENALPE

Section 3.1 p. 33

Forbidden pattern:





☆☆☆☆☆ Can you find the forbidden pattern in the larger grid?

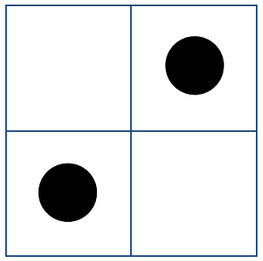
★☆☆☆☆ Find at least three occurrences of the pattern.

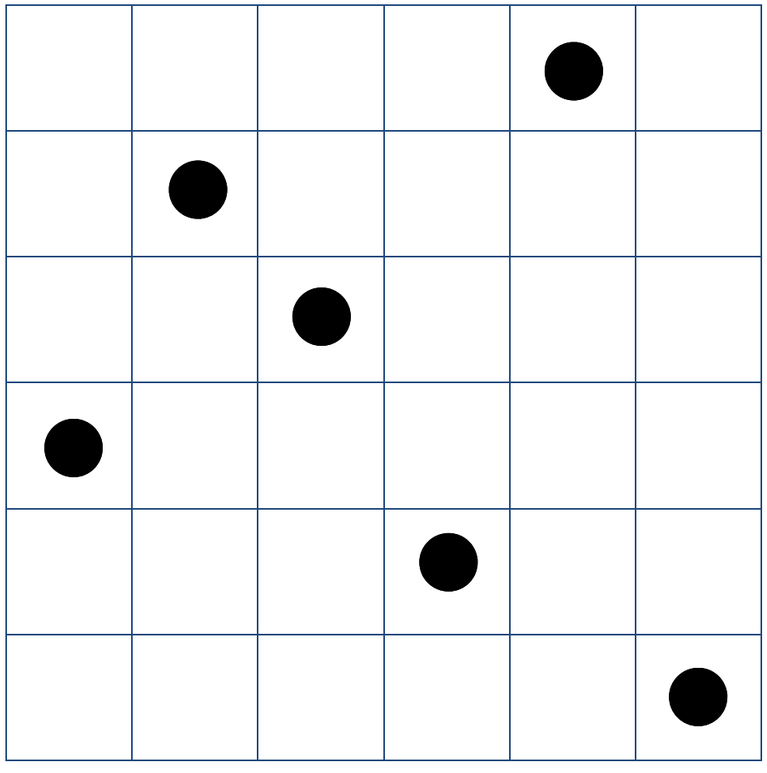
★★☆☆☆ How often does the forbidden pattern appear?

HAMEAU

Section 3.1 p.33

Forbidden pattern:





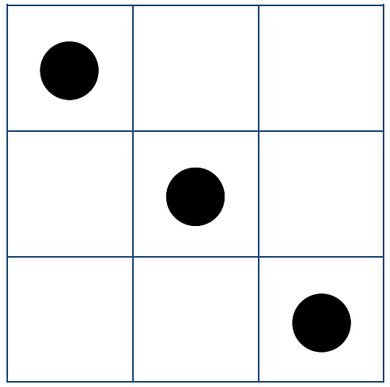
★☆☆☆☆ Find at least three occurrences of the pattern.

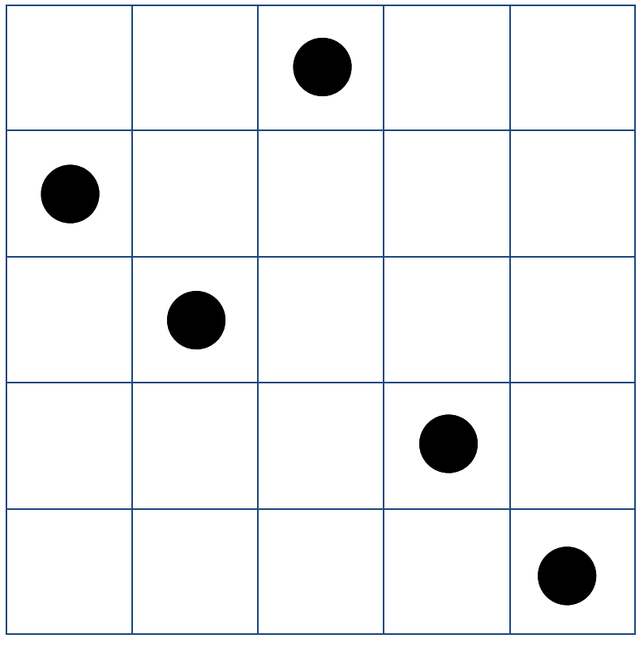
★★☆☆☆ How often does the forbidden pattern appear?

DONAUINSEL

Section 3.1 p.33

Forbidden pattern:





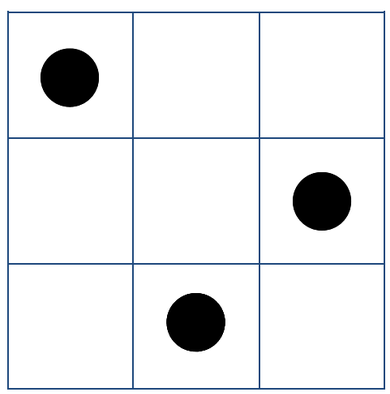
★☆☆☆☆ Does the forbidden pattern appear in the above grid?

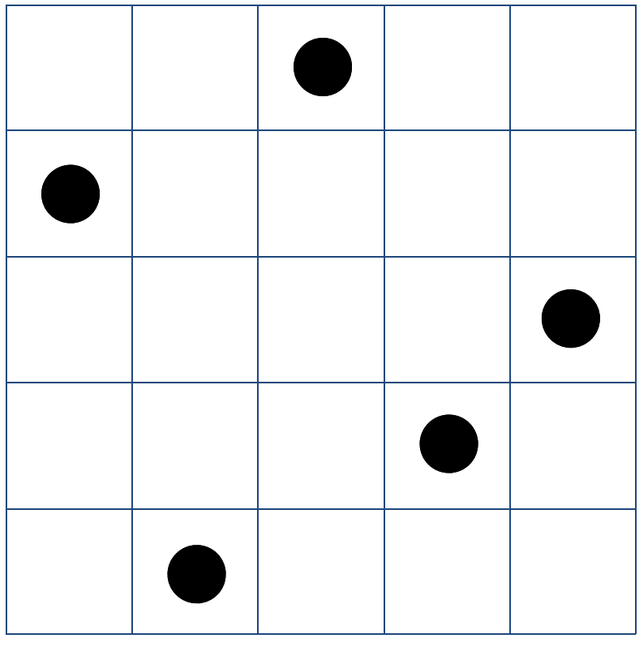
★★☆☆☆ Find all five occurrences of the pattern.

LOBAU

Section 3.1 p.33

Forbidden pattern:





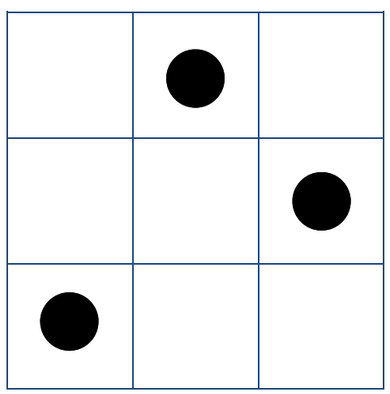
★★☆☆☆ Does the forbidden pattern appear in the above grid?

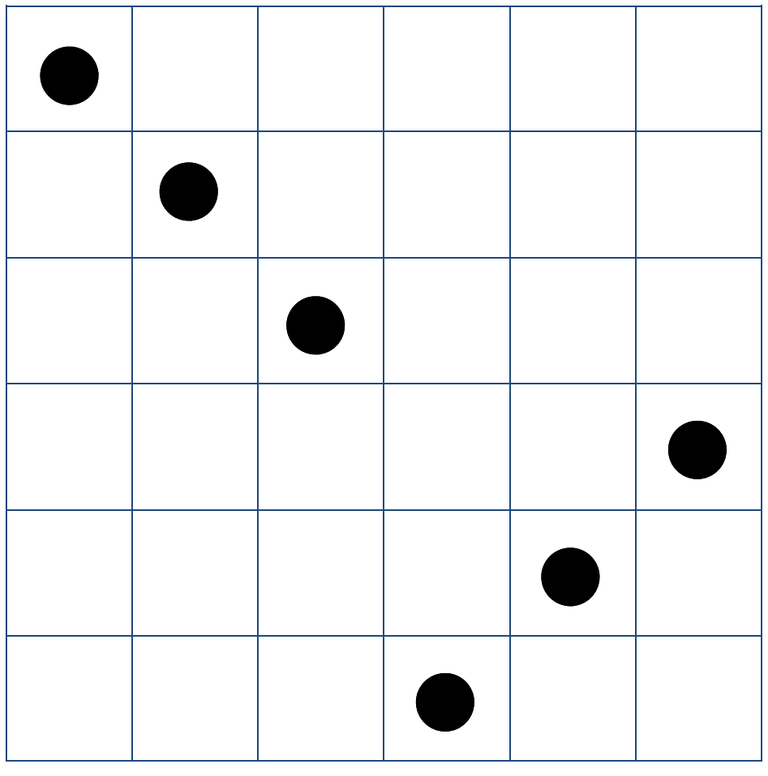
★★★☆☆ If yes, how often does it appear?

MARSWIESE

Section 3.1 p.33

Forbidden pattern:



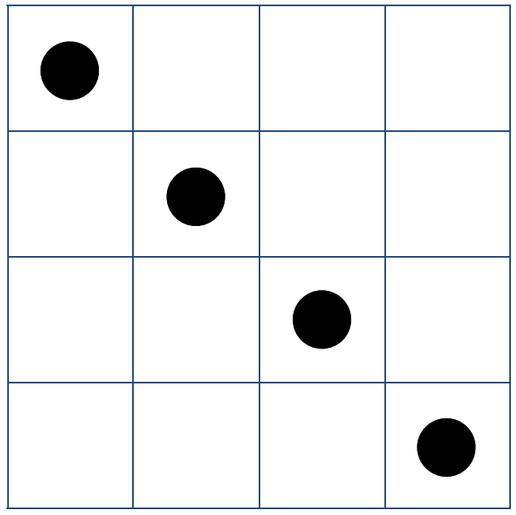


★★★☆☆ Does the forbidden pattern appear in the above grid?

AM HIMMEL

Section 3.1 p.33

Forbidden pattern:

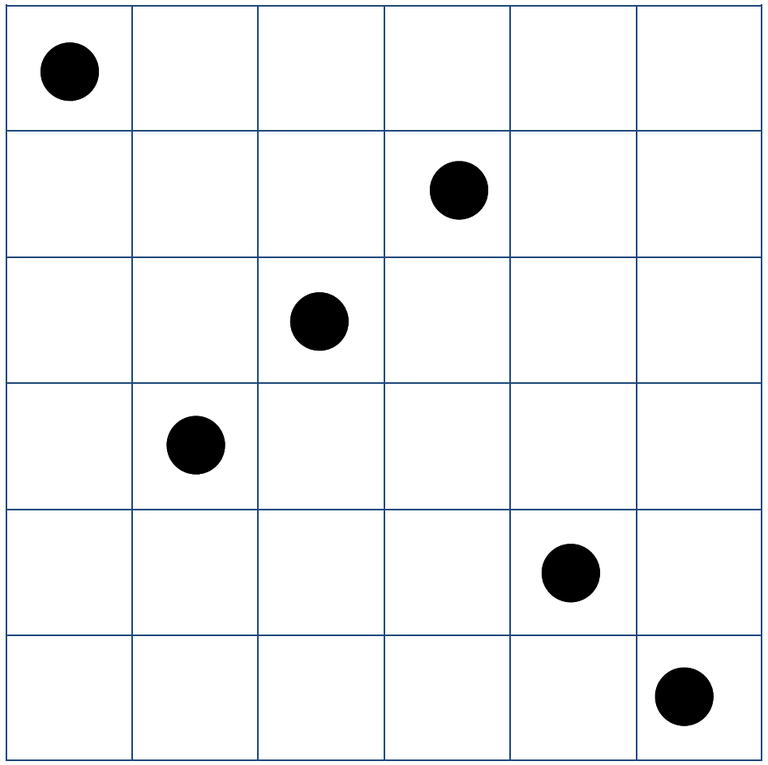


★★☆☆☆

Does the forbidden pattern appear in the grid below?

★★★☆☆

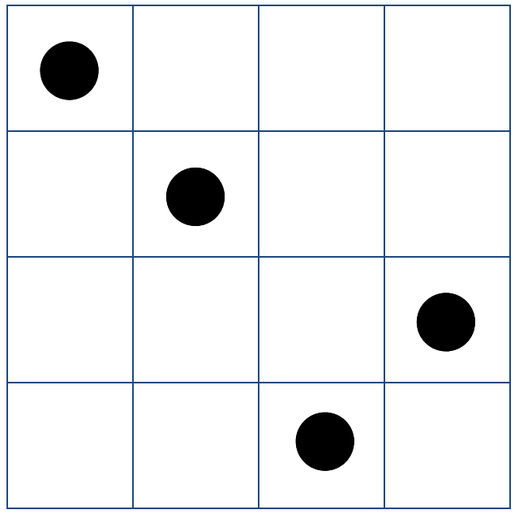
Find all three occurrences of the forbidden pattern.



COBENZL

Section 3.1 p.33

Forbidden pattern:

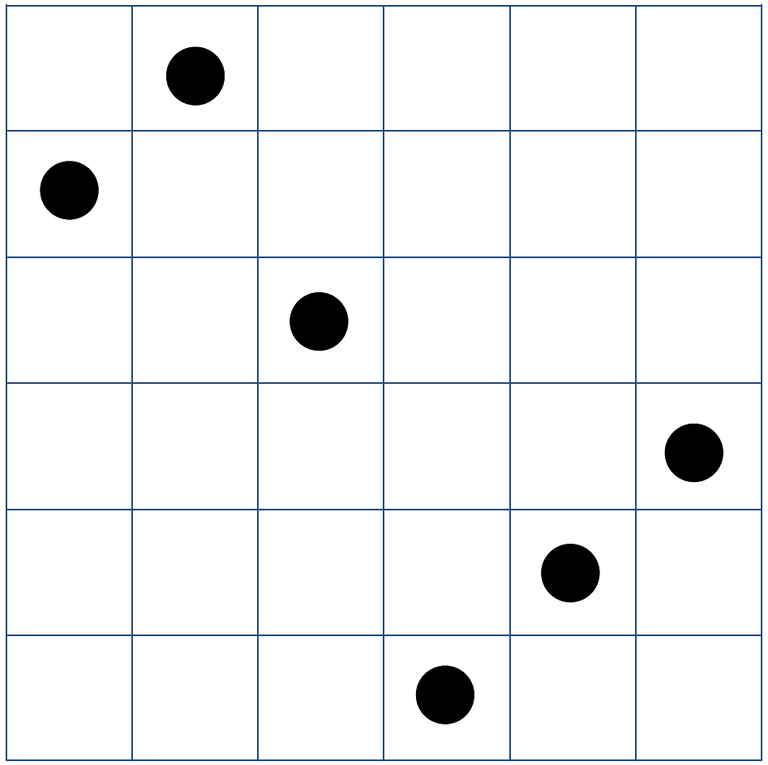


★★☆☆☆

Does the forbidden pattern appear in the grid below?

★★★☆☆

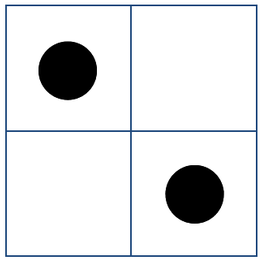
How often does the forbidden pattern occur?

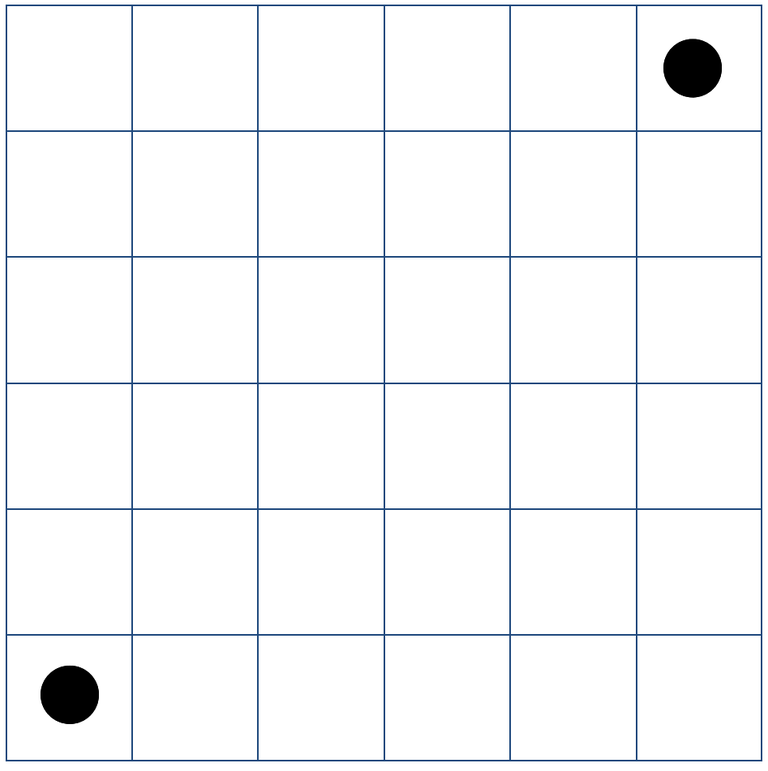


KAHLENBERG

Section 3.1 p.35

Forbidden pattern:





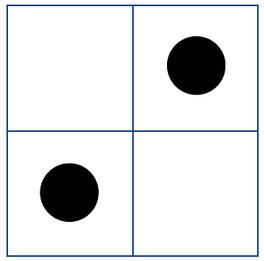
★★☆☆☆ Place the remaining four pawns so that the forbidden pattern doesn‘t appear.

★★★☆☆ How many such possibilities are there?

LEOPOLDSBERG

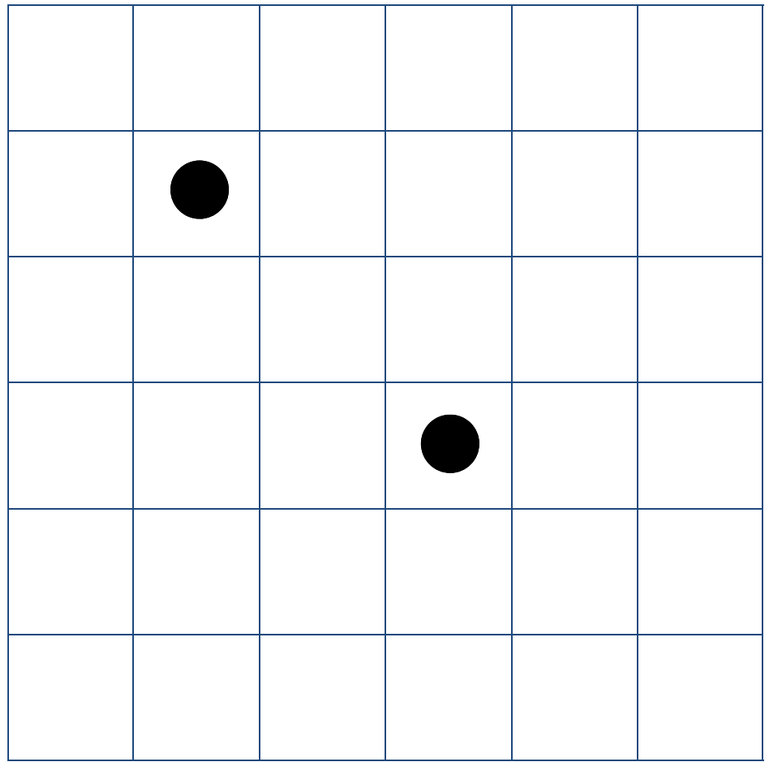
Section 3.1 p.35

Forbidden pattern:



★★☆☆☆

Place the remaining four pawns so that the forbidden pattern doesn‘t appear.



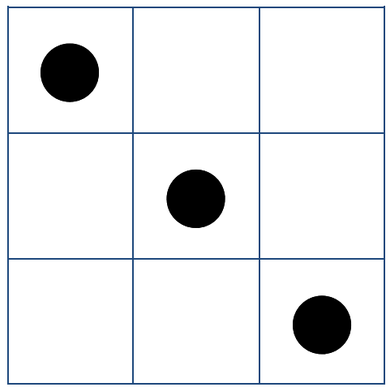
★★★☆☆ How many such possibilities are there?

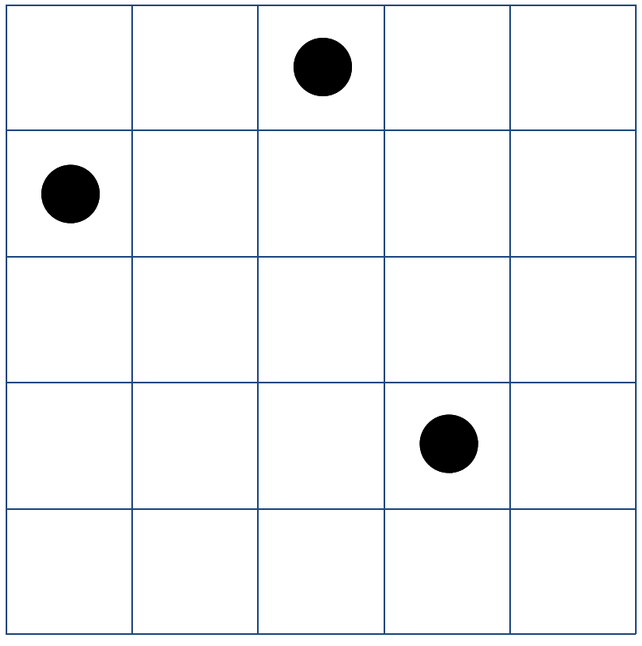
★★★☆☆ Is there a connection between Kahlenberg and Leopoldsberg?

STEPHANIEWARTE

Section 3.2 p. 35

Forbidden pattern:





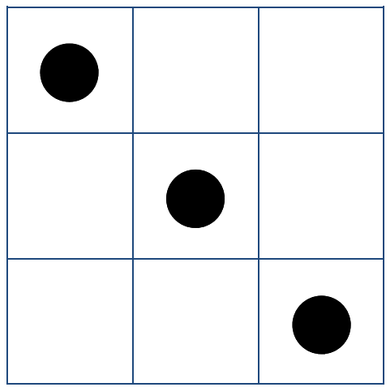
★☆☆☆☆ Place the remaining two pawns so that the forbidden pattern appears.

★★☆☆☆ Can they be placed so that it doesn‘t appear?

HIRSCHENKOGEL

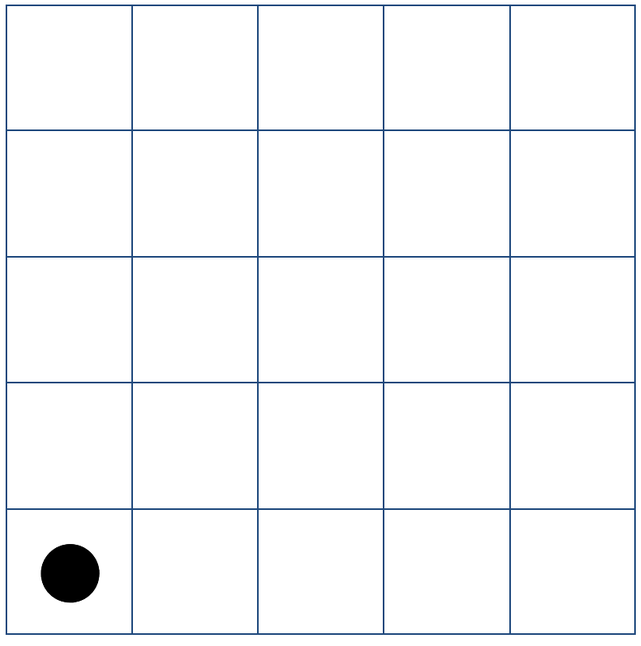
Section 3.2 p. 35

Forbidden pattern:



★☆☆☆☆

Place the remaining four pawns so that the forbidden pattern doesn‘t appear.



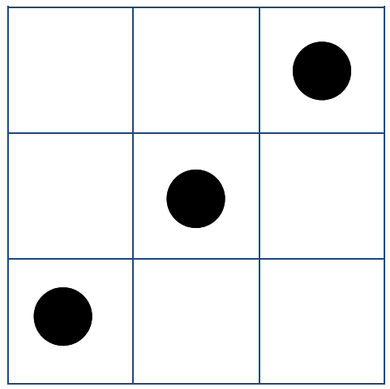
★★☆☆☆ Find two different possibilities of placing the pawns so that the pattern doesn‘t appear.

★★★★★ How many such possibilities are there?

BISAMBERG

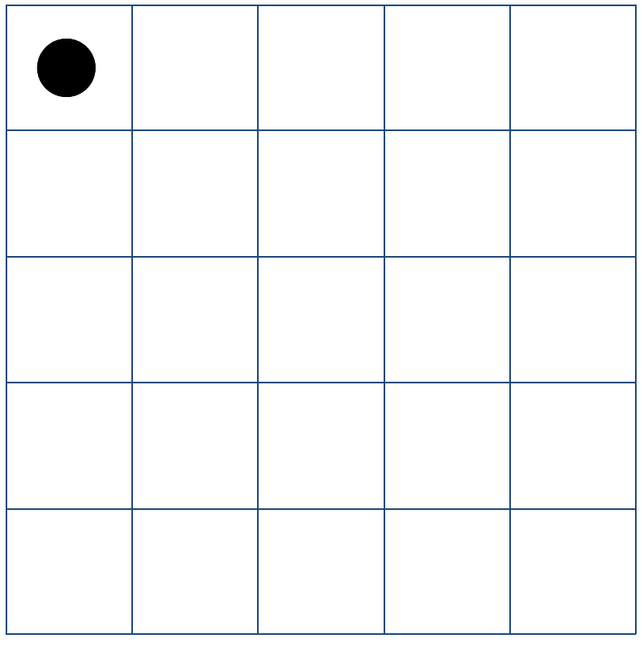
Section 3.2 p.35

Forbidden pattern:



★★☆☆☆

Place the remaining four pawns so that the forbidden pattern doesn‘t appear.



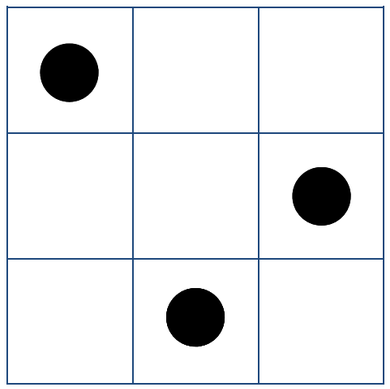
★★★☆☆ Find three different possibilities of placing the pawns so that the pattern doesn‘t appear.

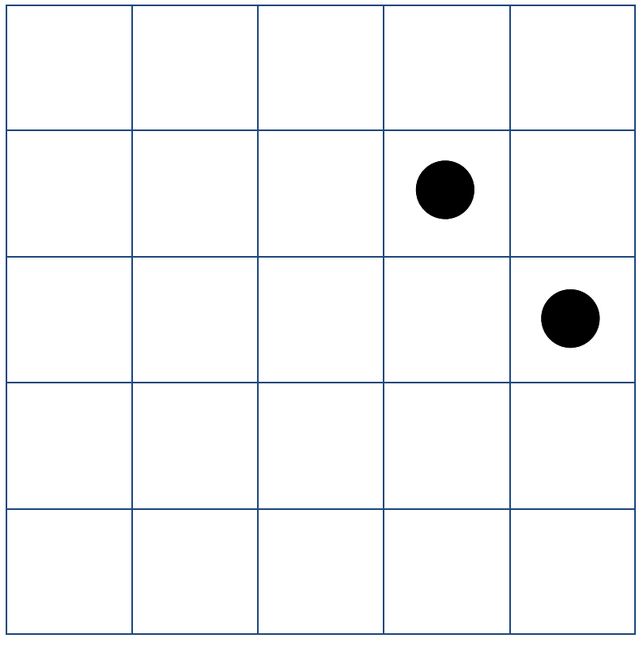
★★★★☆ Can you see a connection between Hirschenkogel and Bisamberg?

HERMANNSKOGEL

Section 3.2 p.35

Forbidden pattern:





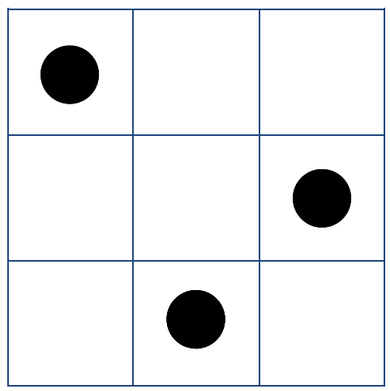
★★☆☆☆ Place the remaining three pawns so that the forbidden pattern appears.

★★★☆☆ Can they be placed so that it doesn‘t appear?

DÜRRENSTEIN

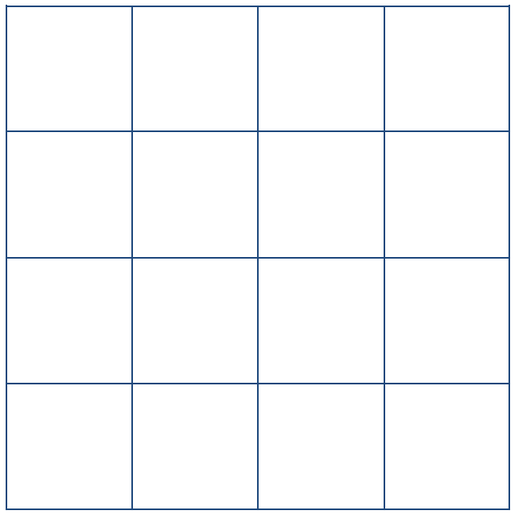
Section 3.2 p.35

Forbidden pattern:



★★☆☆☆

Place the four pawns so that the forbidden pattern doesn‘t appear.



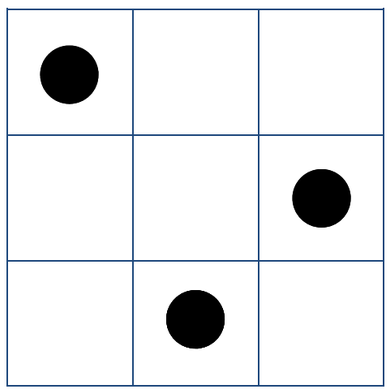
★★★★☆ Find four different possibilities of placing the pawns so that the pattern doesn‘t appear.

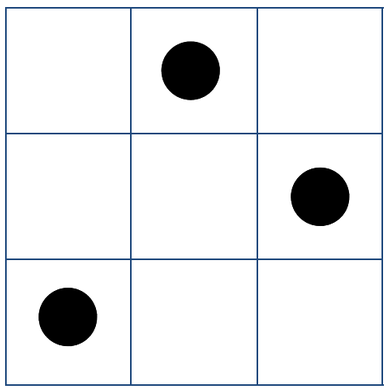
★★★★★ Are there more possibilities of placing the pawns so that the pattern appears or so that it doesn‘t appear?

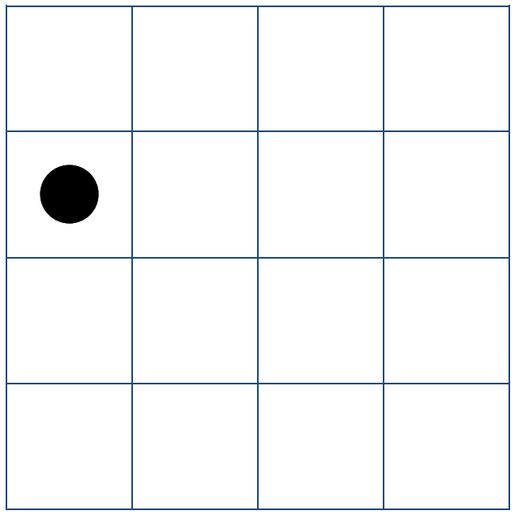
ÖTSCHER

Simion & Schmidt p.41

Forbidden patterns:







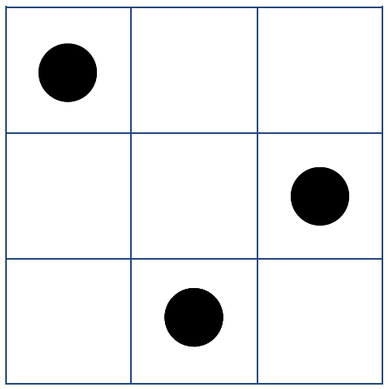
★★★★☆ Place the remaining four pawns so that both forbidden patterns are avoided.

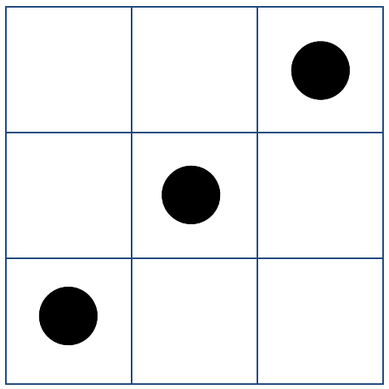
★★★★☆ Can the four pawns be placed in a way that both forbidden patterns occur?

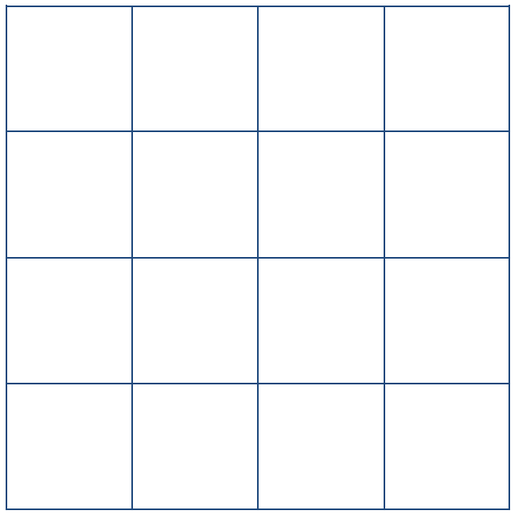
JAUERLING

Simion & Schmidt p.41

Forbidden patterns:







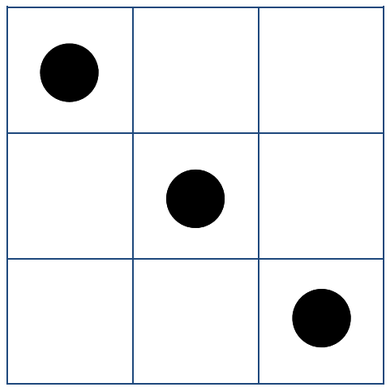
★★★☆☆ Place the five pawns so that both forbidden patterns are contained.

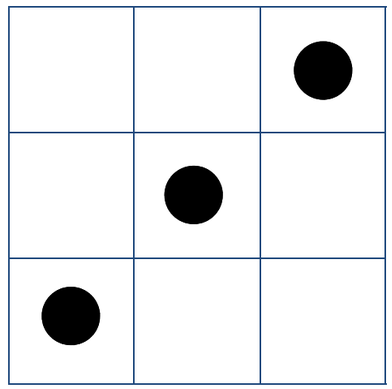
★★★★☆ Find four different possibilities of placing the five pawns so that both patterns are avoided.

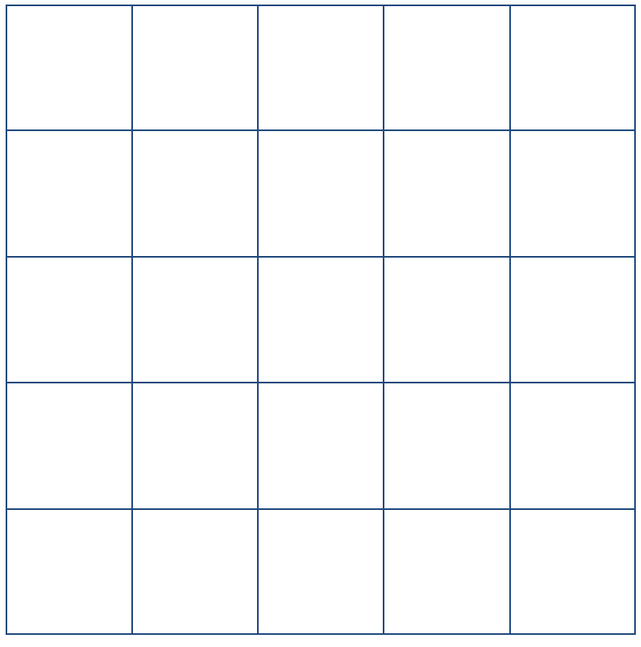
PLANSPITZE

Erdős-Szekeres p.41

Forbidden patterns:







★★★★☆

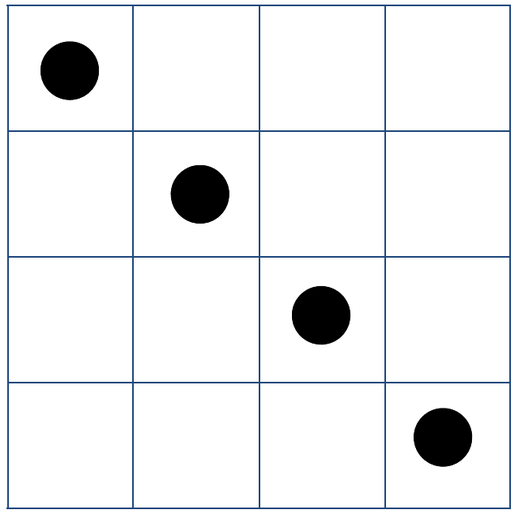
Place the five pawns so that both forbidden patterns are contained.

★★★★★ How many possibilities are there of placing five pawns so that both forbidden patterns are avoided?

HOHE WAND

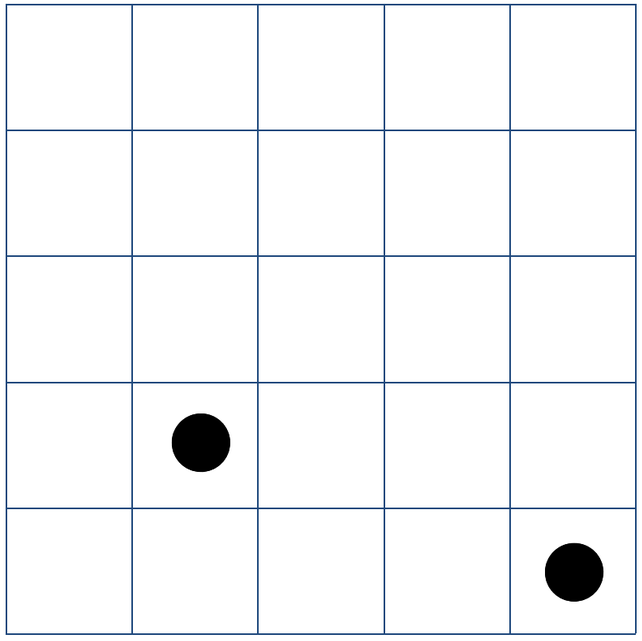
Section 3.3 p.43

Forbidden pattern:



★★☆☆☆

Place the remaining three pawns so that the forbidden pattern is avoided.

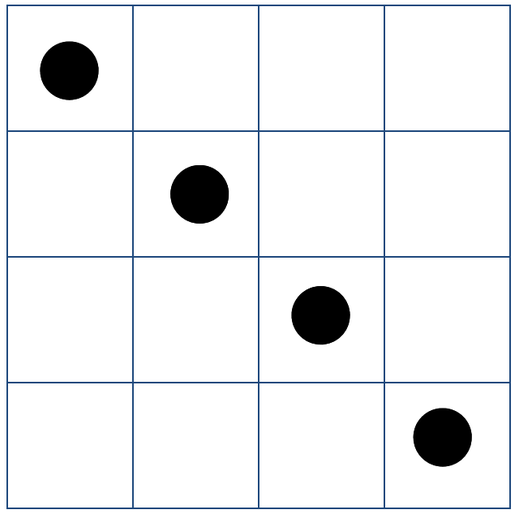


★★★☆☆ How many possibilities are there to place the pawns so that the pattern is contained?

RAXALPE

Section 3.3 p.43

Forbidden pattern:

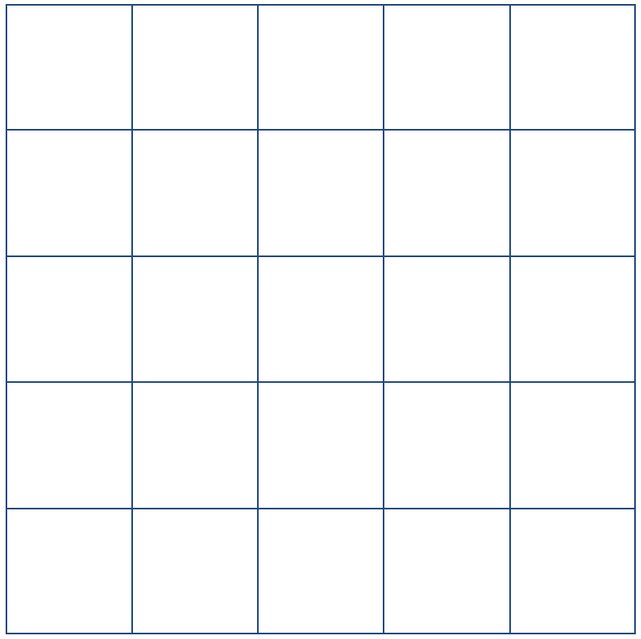


★★☆☆☆

Place the five pawns so that the forbidden pattern is avoided.

★★★★★

How many such possibilities are there?

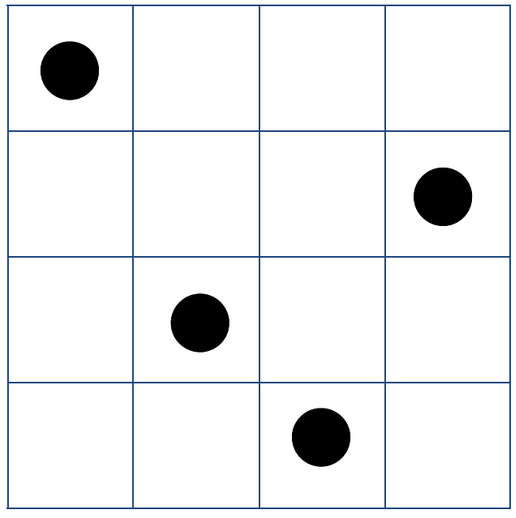


★★★★★ Is it easier to avoid or to contain the pattern?

PEILSTEIN

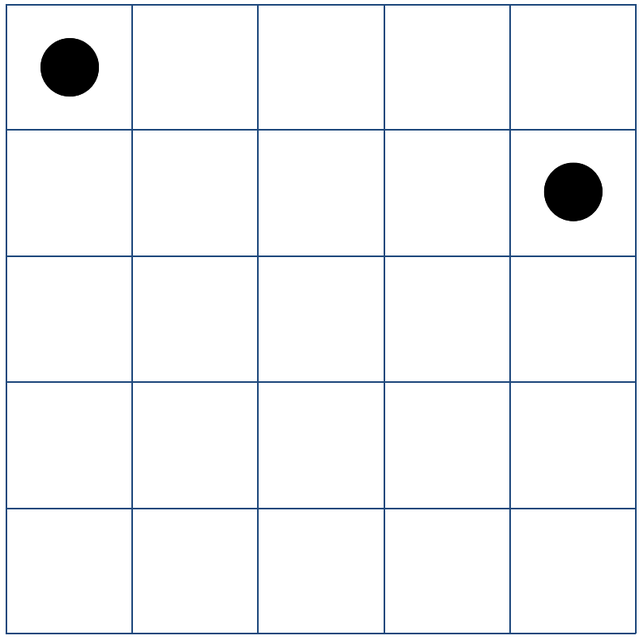
Section 3.3 p.44

Forbidden pattern:



★★☆☆☆

Place the remaining three pawns so that the forbidden pattern is contained.

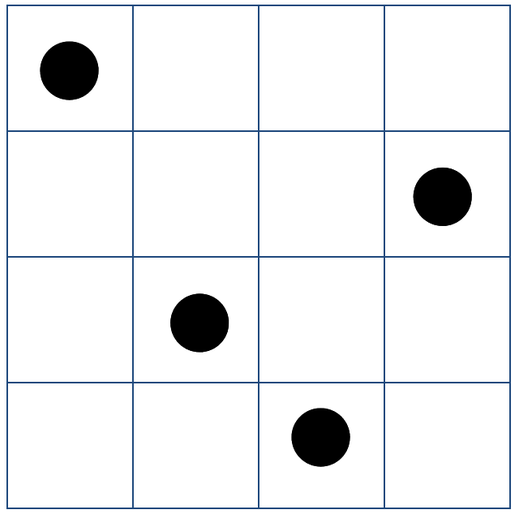


★★★☆☆ Can they be placed so that it is avoided?

HOHE MANDLING

Section 3.3 p.44

Forbidden pattern:

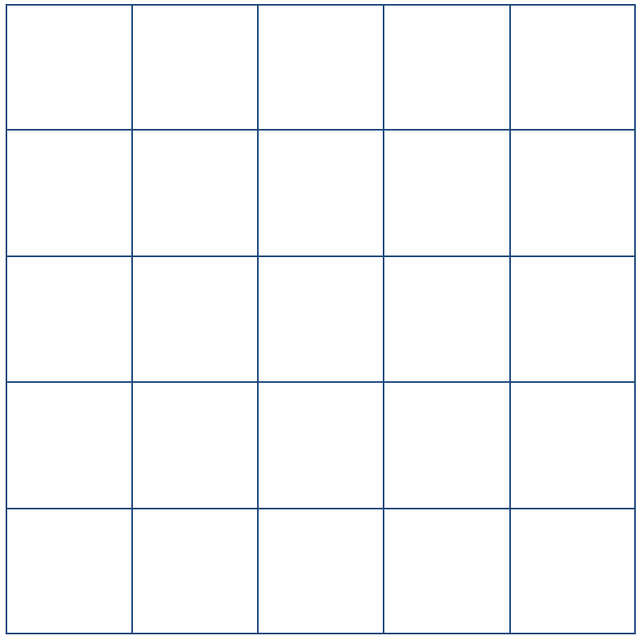


★★☆☆☆

Place the five pawns so that the forbidden pattern is avoided.

★★★★★

How many such possibilities are there?

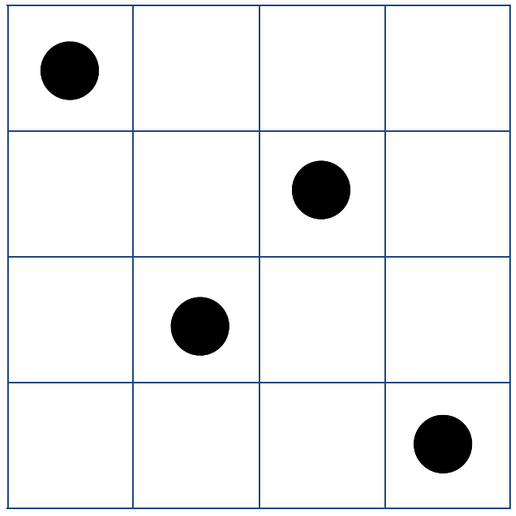


★★★★★ Is it easier to avoid Raxalpe or H. Mandling?

DÜRRE WAND

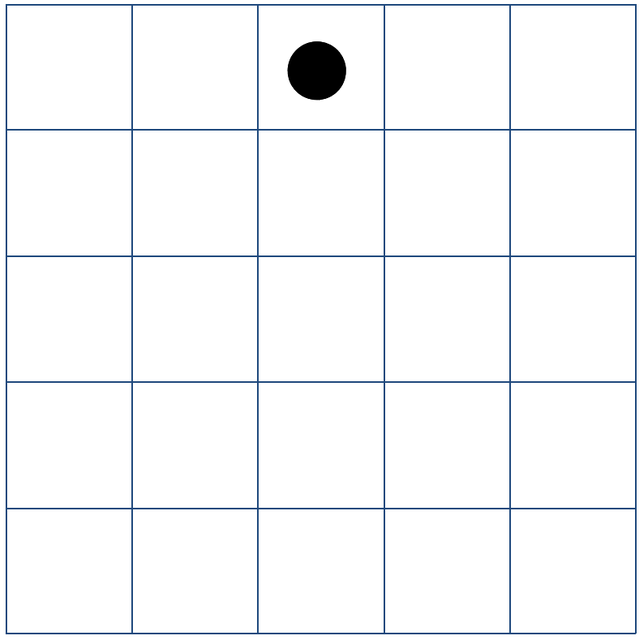
Section 3.3 p.45

Forbidden pattern:



★★☆☆☆

Place the remaining four pawns so that the forbidden pattern is contained.

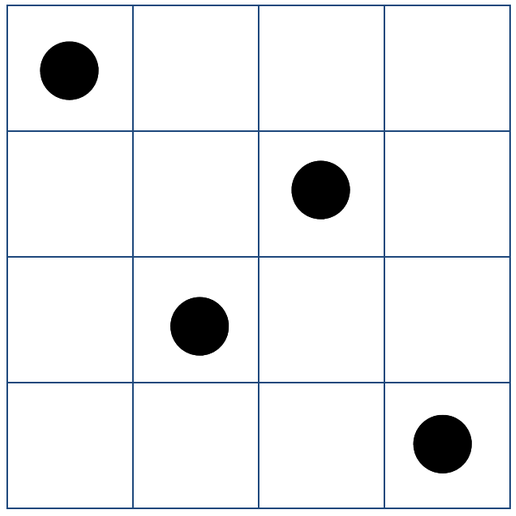


★★★☆☆ Can they be placed so that it is avoided?

SCHNEEBERG

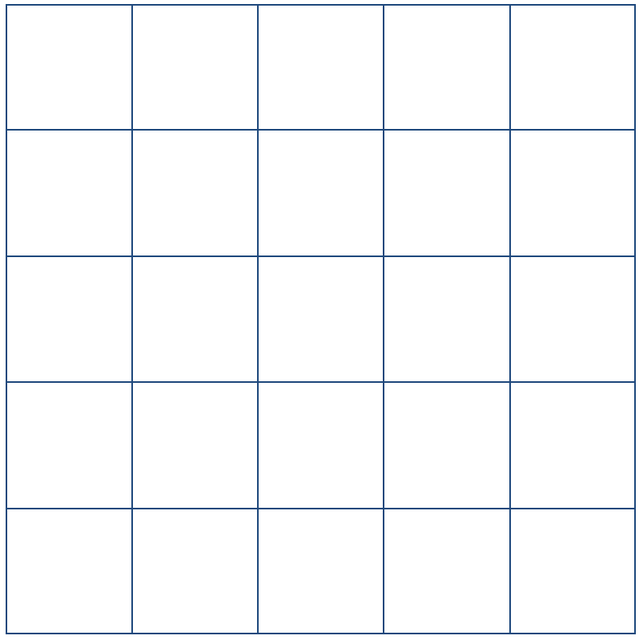
Section 3.3 p.45

Forbidden pattern:



★★★★★

In how many ways can the five pawns be placed so that the forbidden pattern is avoided?



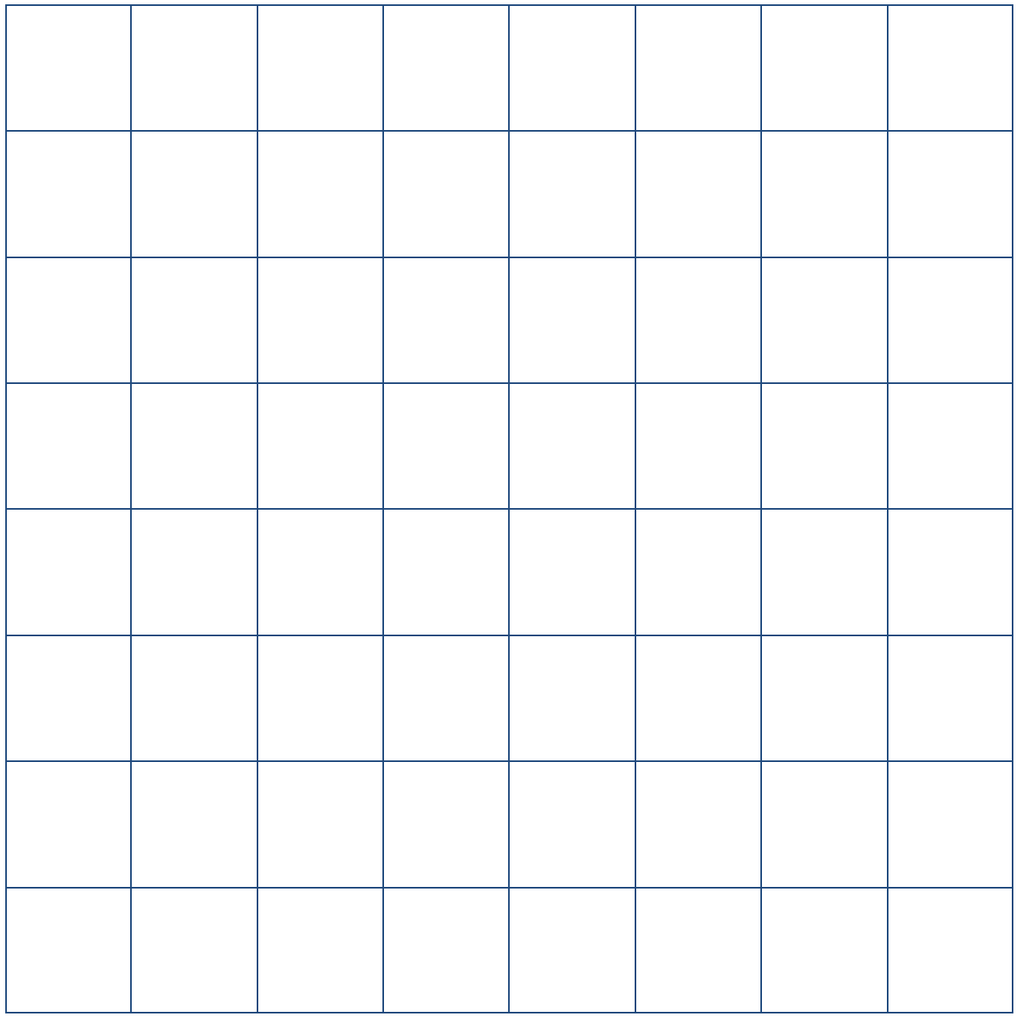
★★★★★ Is it easier to avoid Raxalpe or Schneeberg?

HÖLLENTAL

Section 3.1 p.33

Chapters 2, 4 & 5

Fill in eight pawns in an allowed way.



★☆☆☆☆ Is any pattern of size 2x2 avoided?

★★☆☆☆ Find a pattern of size 3x3 that is contained.

★★★☆☆ Find a pattern of size 3x3 that is avoided.

Part 2

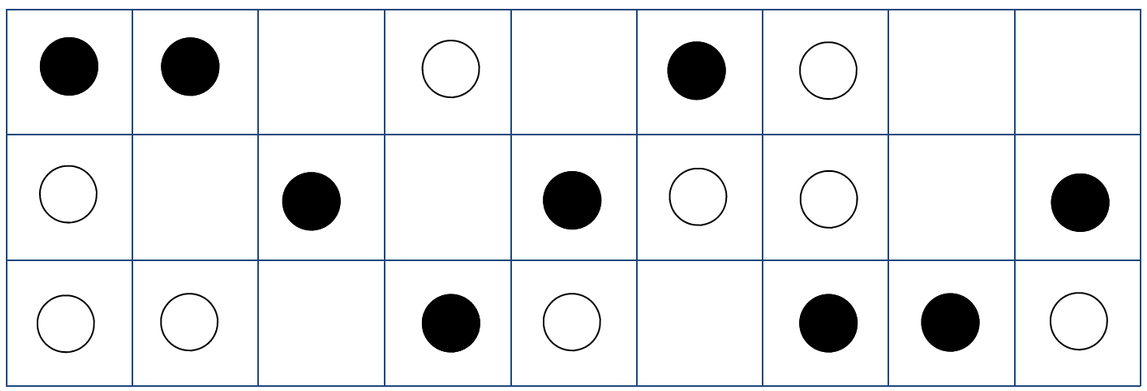
RECTANGULAR GRIDS

(Permutations on regular multisets)

RULES

Preliminaries p. 20

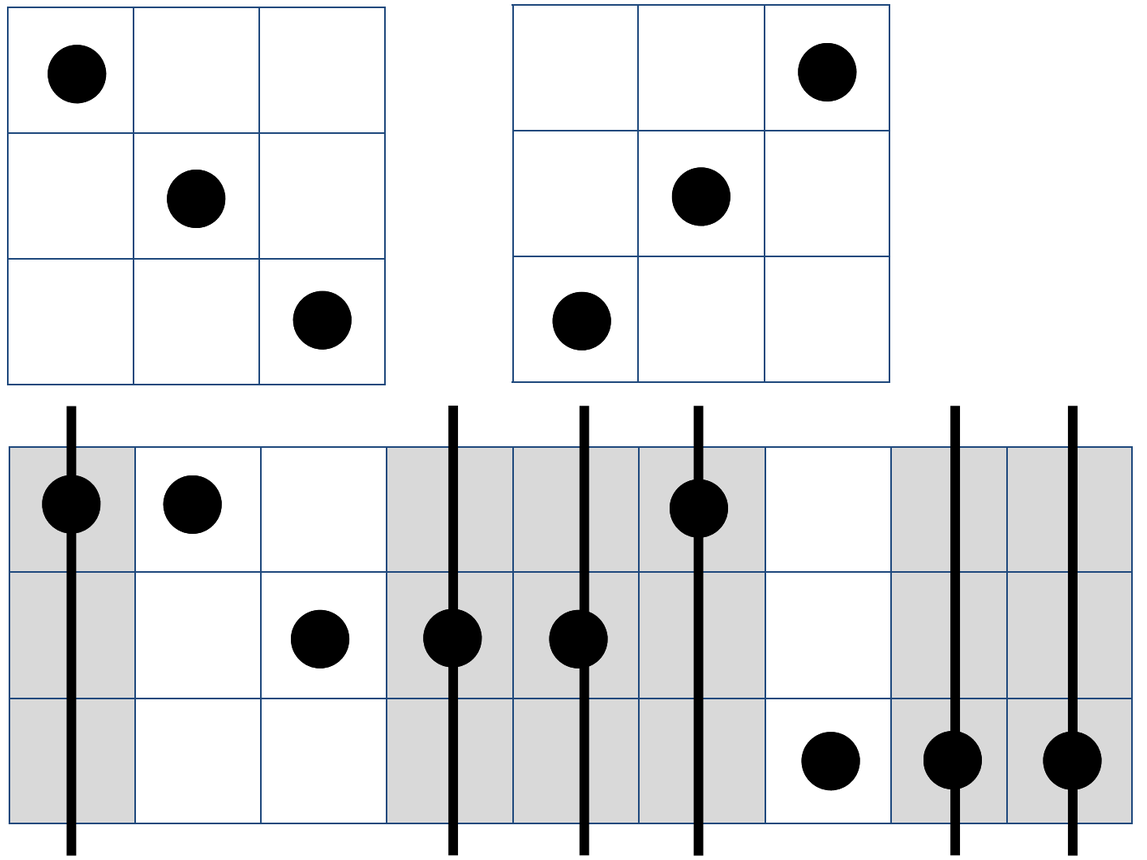
These riddles are similar to those in part 1 but we now have rectangular grids instead of square ones. In these grids, we place as many pawns as there are columns. In the example below we thus place nine pawns. These pawns have to be placed in the empty spaces following a more complicated rule. In every column there has to be exactly one pawn. In every row there has to be the exact same number of pawns. This number of pawns per row can be calculated easily: divide the number of columns by the number of rows. In our example, this means that there have to be 9/3=3 pawns in every row. For instance, placing the pawns like the black ones would be allowed but placing them like the white ones would be forbidden.



For those who like numbers: the black configuration corresponds to the permutation 332123112. And for those who like complicated words: these permutations are called permutations on *regular multisets*.

Chapter 4 p. 55

As in the first part, we say that a pattern corresponding to a smaller grid like the 3x3-grids below is contained in a larger grid if, by deleting some rows and some columns, we obtain a grid in the size of the pattern that looks exactly like the pattern.

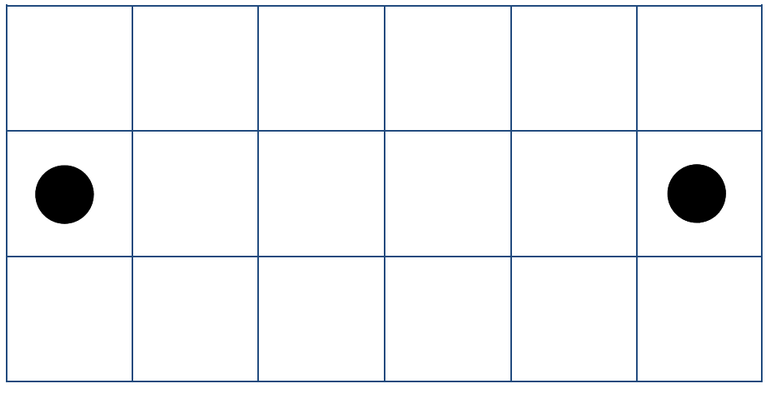


In this example, we can see that the 3x9-grid contains the pattern on the left by deleting the columns marked in grey. No rows may be deleted. This is not possible for the pattern on the right. Therefore we say that the pattern on the right is avoided. We say: 332223111 contains 321 but avoids 123.

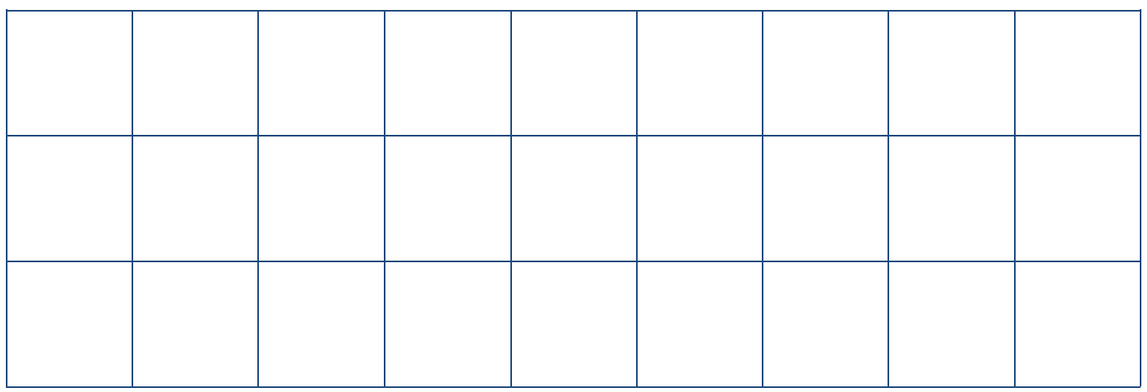
BURGGARTEN

Preliminaries p. 20

Forbidden pattern: none



★☆☆☆☆ Place the remaining four pawns in an allowed way.

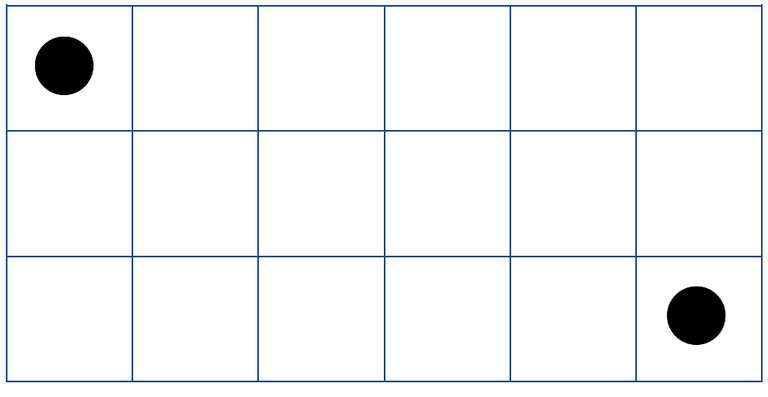


★☆☆☆☆ Place nine pawns in an allowed way.

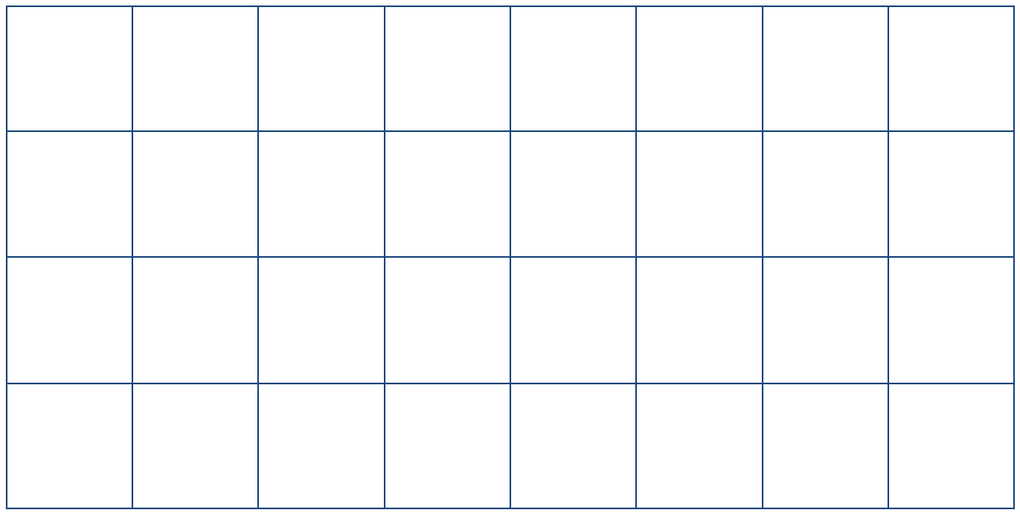
LAAER BERG

Preliminaries p. 20

Forbidden pattern: none



★☆☆☆☆ Find four different ways of placing the remaining four pawns in an allowed way.



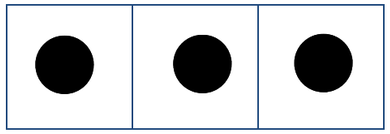
★★☆☆☆ Find five ways of placing eight pawns in an allowed way.

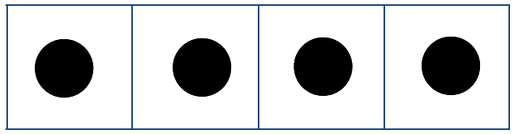
★★★★☆ How many such possibilities are there?

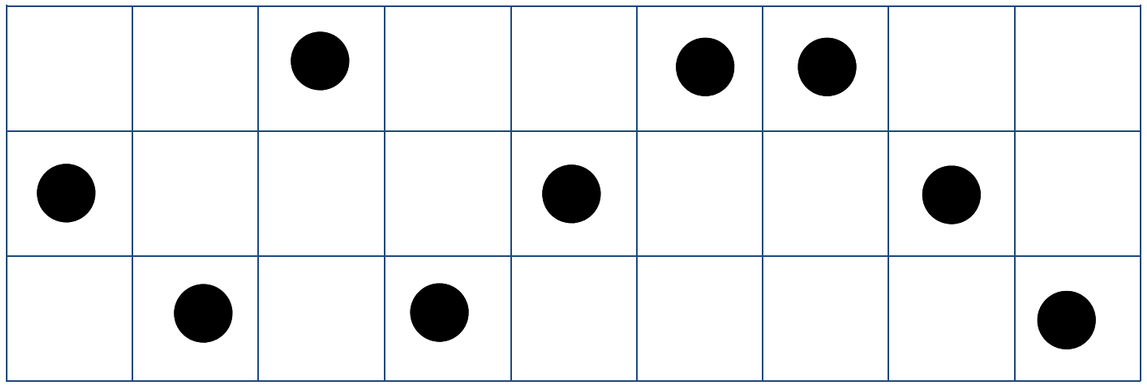
GAHNS

Chapter 4 p. 55

Forbidden patterns:







★☆☆☆☆ Can you find the forbidden pattern on the left hand side in the grid above?

★★☆☆☆ How often is this pattern contained? Find all occurrences.

★★★☆☆ Could the pawns have been placed so that the pattern is avoided?

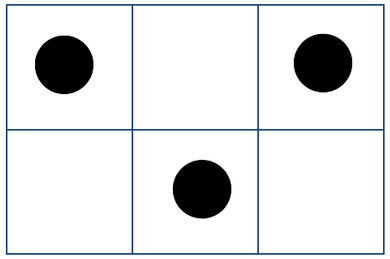
★★☆☆☆ Can you find the pattern on the right?

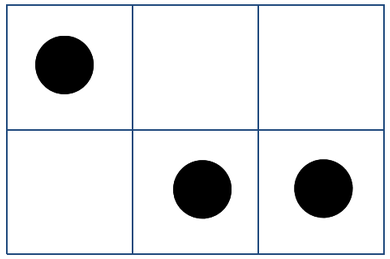
★★★☆☆ Could the pawns have been placed so that it does appear?

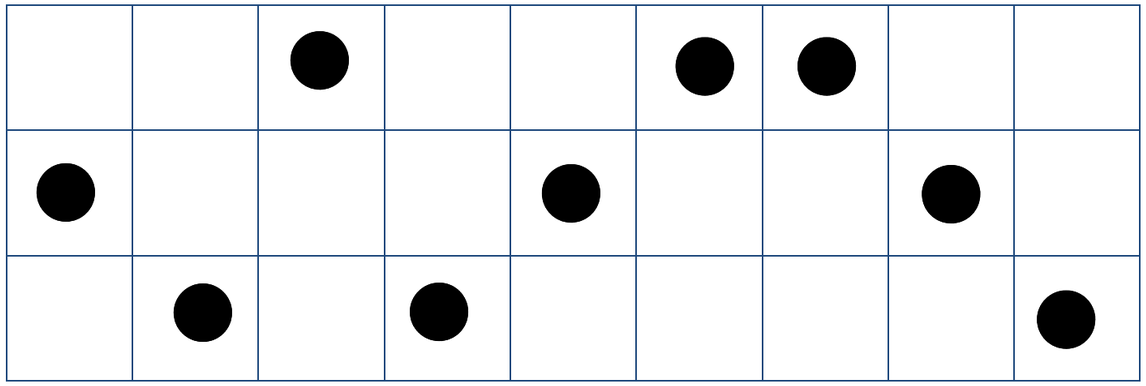
SCHÖPFL

Chapter 4 p. 55

Forbidden patterns:







★★☆☆☆ Can you find the pattern on the left hand side in the grid above?

★★★☆☆ Find at least four occurrences of this pattern.

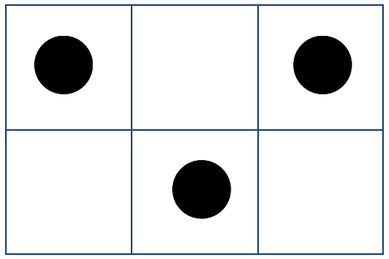
★★☆☆☆ Can you find the pattern on the right hand side in the grid above?

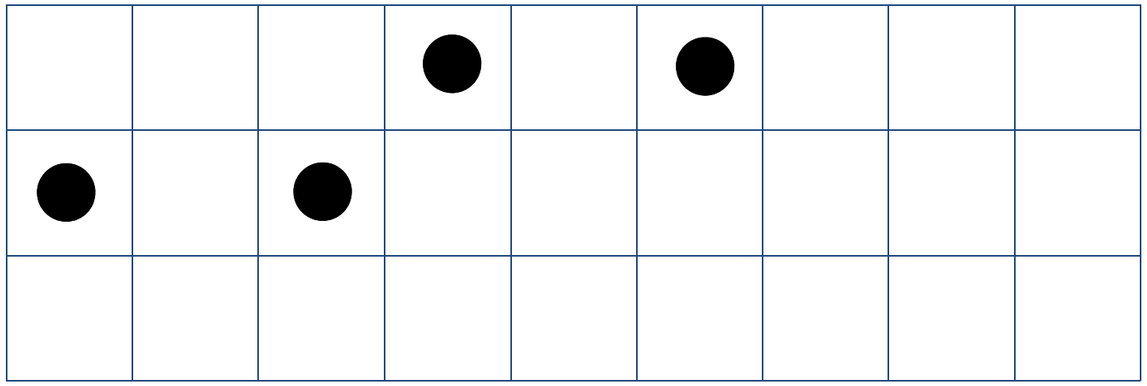
★★★☆☆ Find all occurrences of this pattern.

GESCHRIEBENSTEIN

Chapter 4 p. 55

Forbidden patterns:





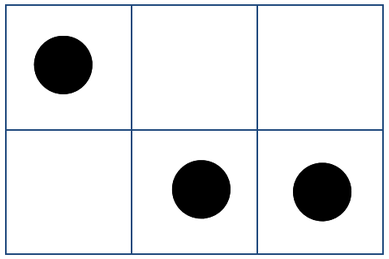
★★☆☆☆ Place the remaining five pawns so that the forbidden pattern is contained.

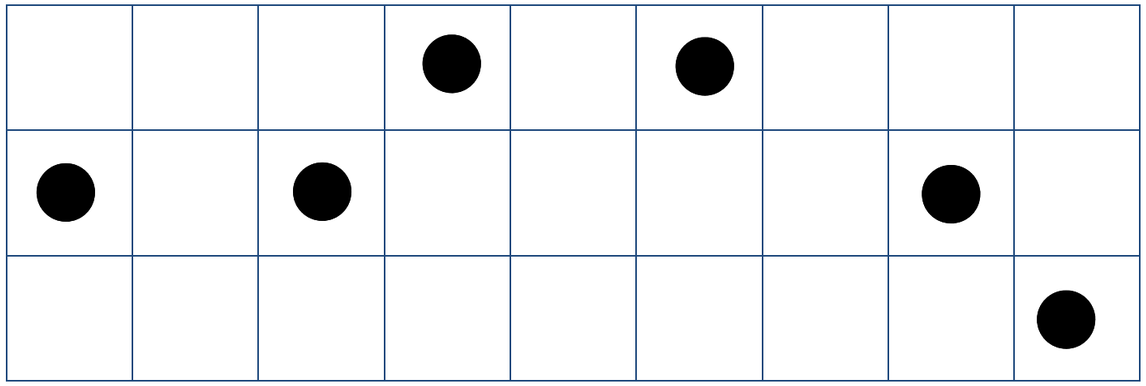
★★★☆☆ Can they be placed in a way so that it doesn‘t appear?

LINDKOGEL

Chapter 4 p. 55

Forbidden patterns:





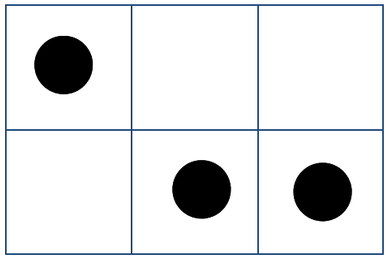
★★☆☆☆ Place the remaining three pawns so that the forbidden pattern is contained.

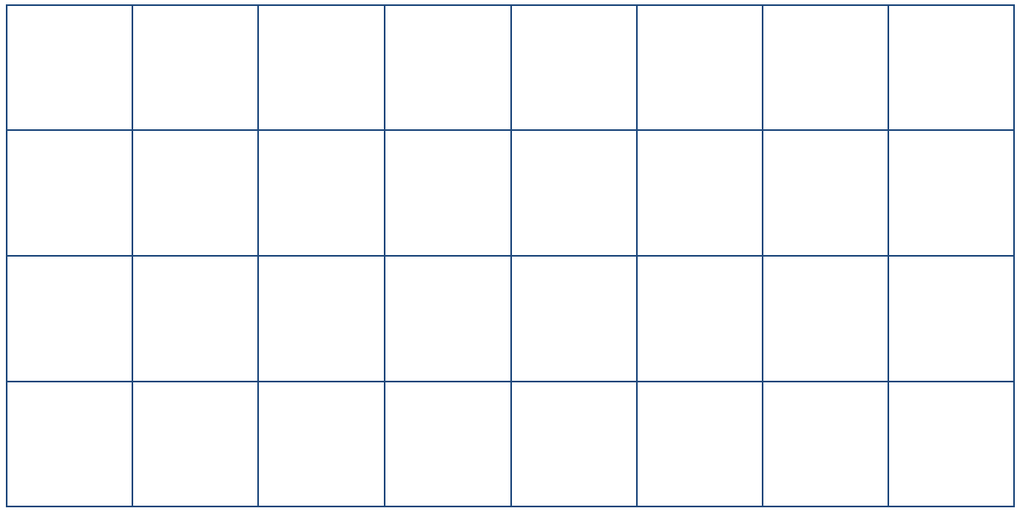
★★★☆☆ Can they be placed in a way so that it doesn’t appear?

HOCHKAR

Section 4.3.1 p. 63

Forbidden patterns:





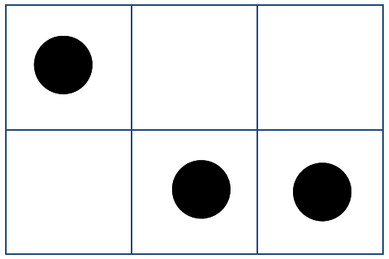
★★★☆☆ Place the eight pawns so that the forbidden pattern is avoided.

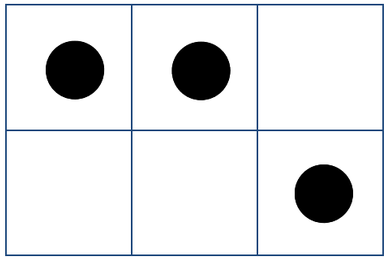
★★★★☆ How many such possibilities are there in total?

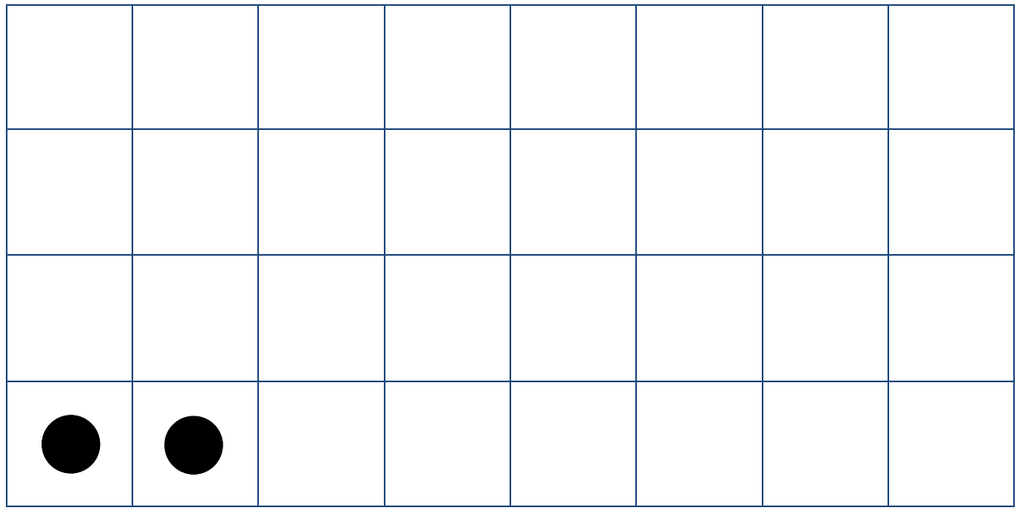
HOHE VEITSCH

Section 4.4 p. 65

Forbidden patterns:







★★★☆☆ Place the remaining six pawns so that both forbidden patterns are avoided.

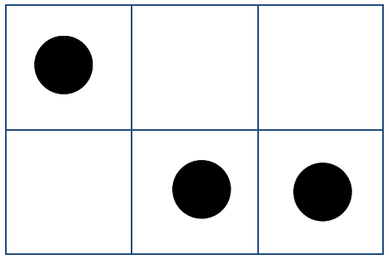
★★★★☆ Find all five possibilities of placing the pawns in such a way.

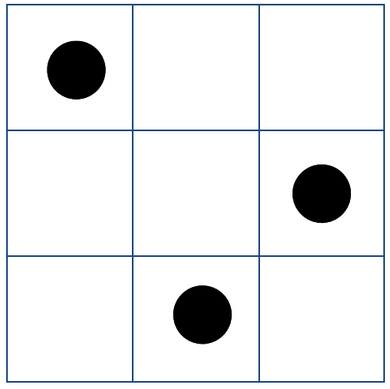
★★★★★ How many possibilities are there if the two initial pawns can be moved somewhere else?

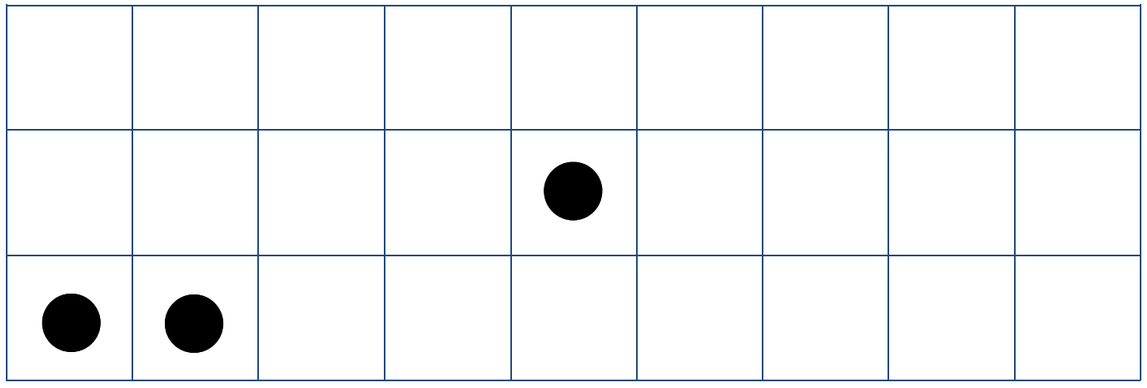
DACHSTEIN

Sections 5.2 and 5.3 p. 73

Forbidden patterns:







★★★☆☆ Place the remaining six pawns so that both forbidden patterns are avoided.

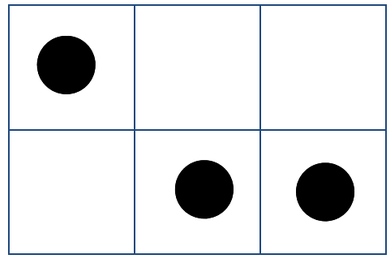
★★★★☆ How many such possibilities are there in total?

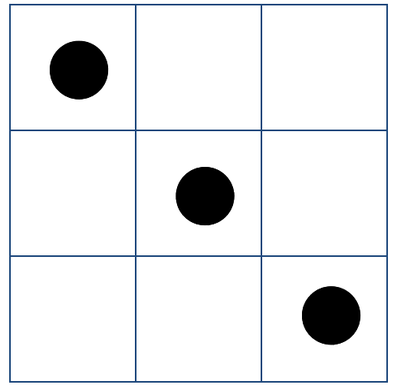
★★★★★ How many possibilities are there if the three initial pawns can be moved somewhere else?

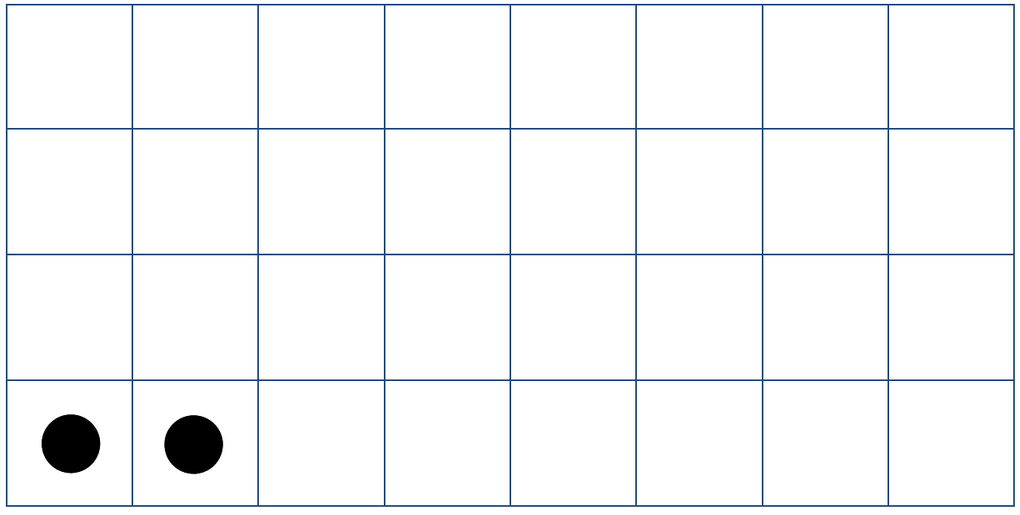
HOCHKÖNIG

Sections 5.2 and 5.3 p. 73

Forbidden patterns:







★★★★☆ Place the remaining six pawns so that both forbidden patterns are avoided.

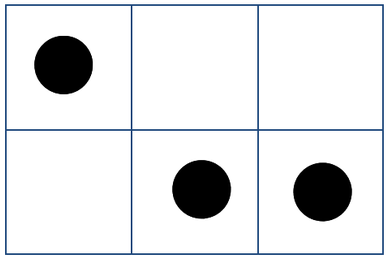
★★★★★ Find six different such possibilities.

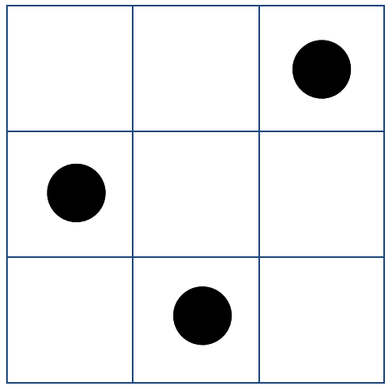
★★★★★ How many possibilities are there if the pawn in the first column can be moved as well?

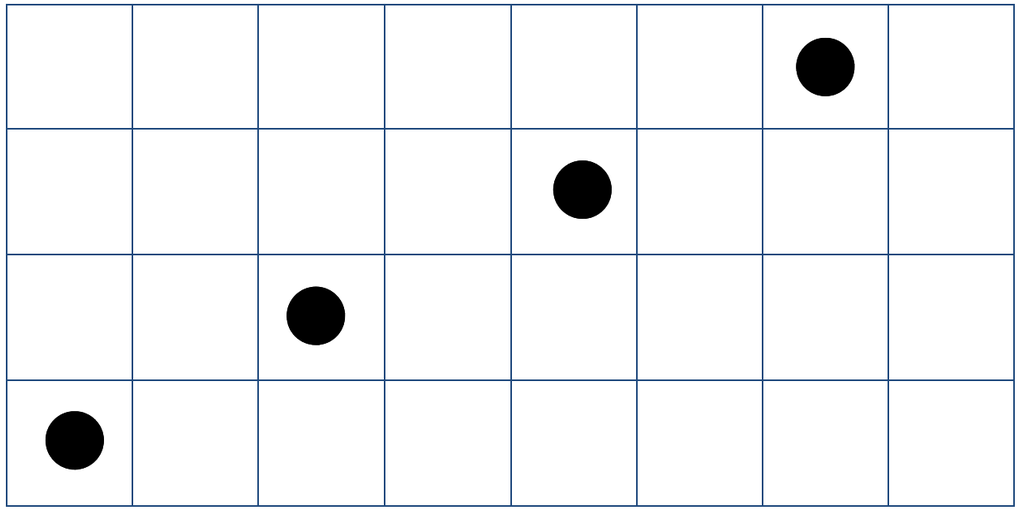
PIZ BUIN

Section 5.4 p. 84

Forbidden patterns:







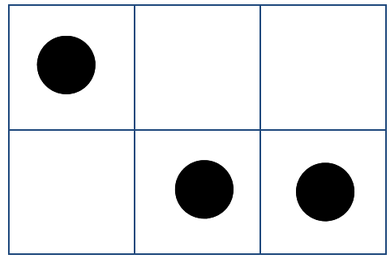
★★★★☆ Place the remaining four pawns so that both forbidden patterns are avoided.

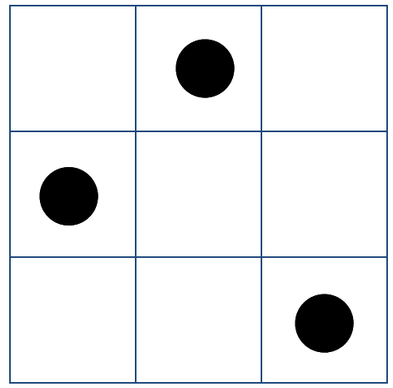
★★★★★ How many such possibilities are there in total?

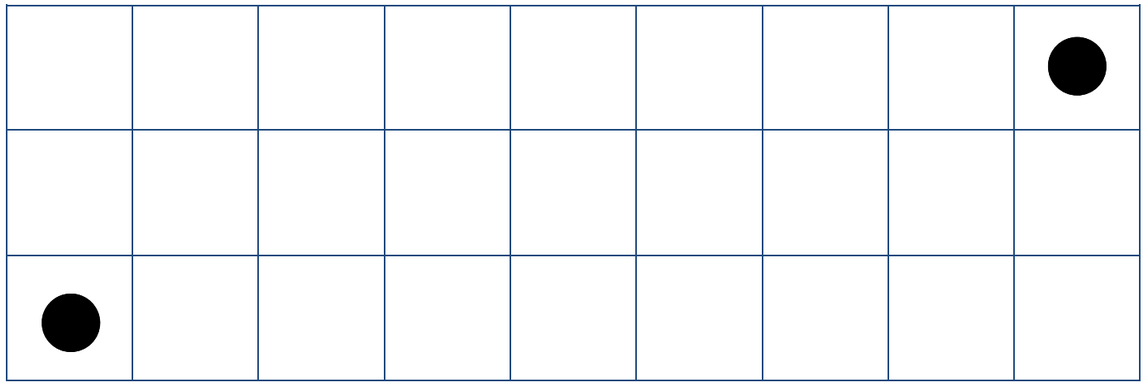
ZUGSPITZE

Section 5.5 p. 90

Forbidden patterns:







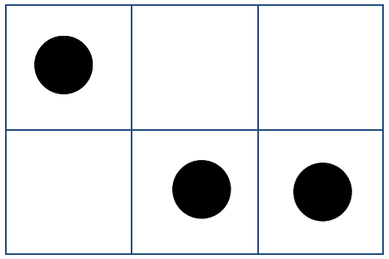
★★★★☆ Place the remaining seven pawns so that both forbidden patterns are avoided.

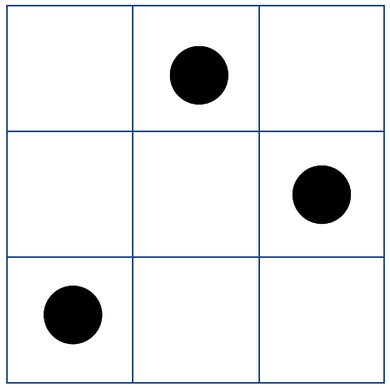
★★★★★ How many such possibilities are there in total?

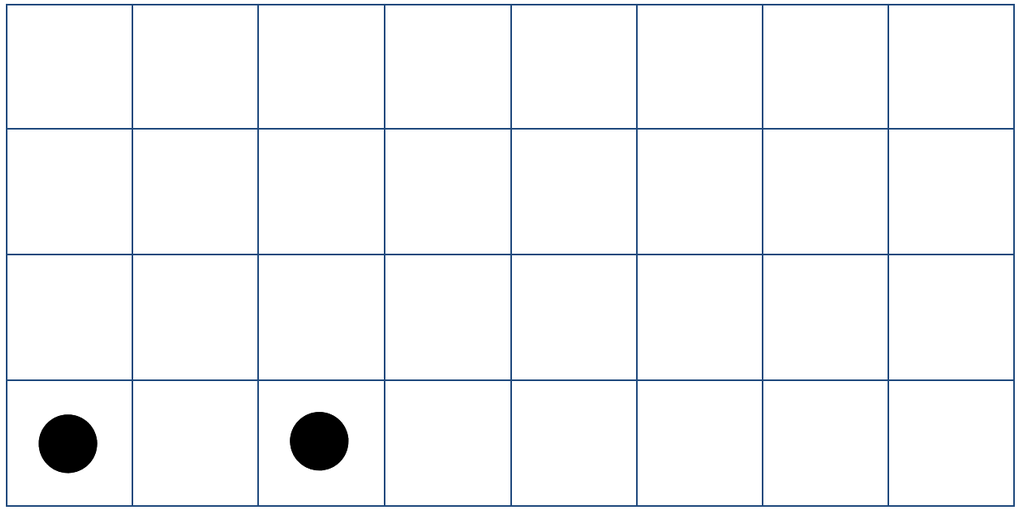
GROßVENEDIGER

Section 5.6 p. 93

Forbidden patterns:







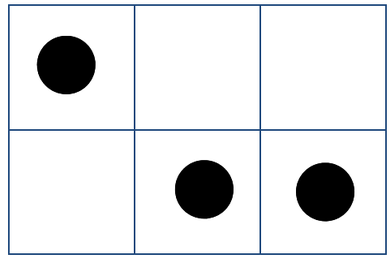
★★★★☆ Place the remaining six pawns so that both forbidden patterns are avoided.

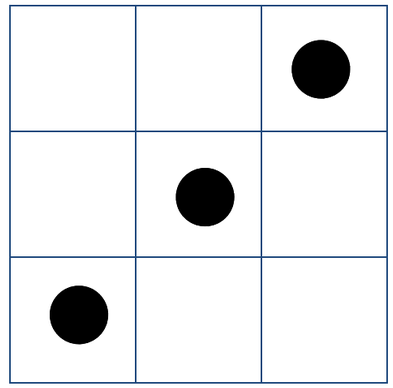
★★★★★ How many such possibilities are there in total?

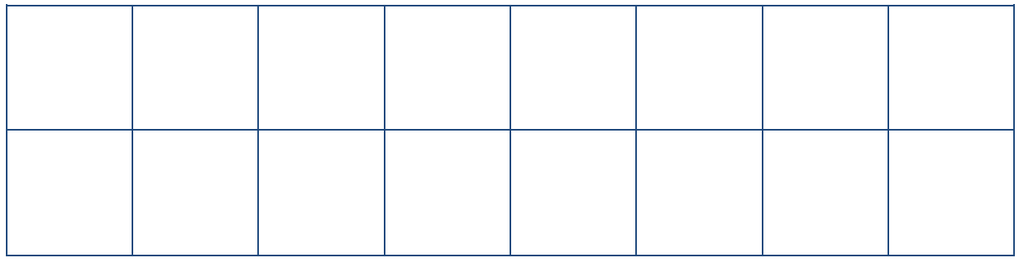
GROßGLOCKNER

Section 5.7 p. 94

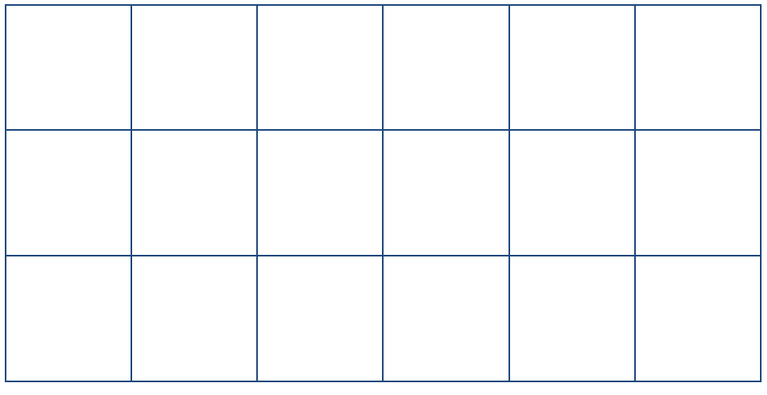
Forbidden patterns:







★★★★☆ Place the pawns so that both forbidden patterns are avoided.

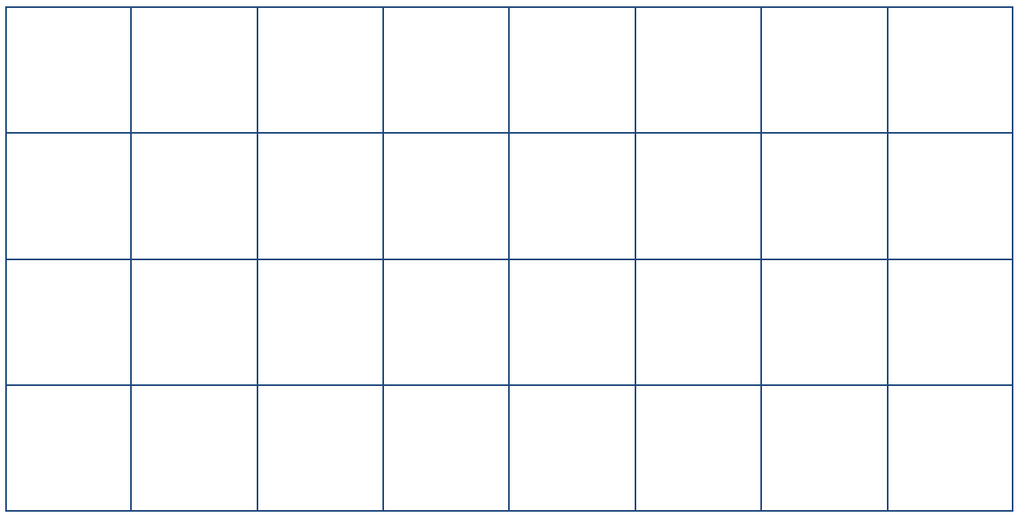


★★★★☆ Can the pawns be placed so that both patterns are avoided?

STUHLECK

Section 4.1 p. 55

Fill in eight pawns in an allowed way.



★☆☆☆☆ Is any pattern of size 1x2 avoided?

★☆☆☆☆ Is any pattern of size 1x3 avoided?

★☆☆☆☆ Is any pattern of size 2x2 avoided?

★★☆☆☆ Find a pattern of size 3x3 that is contained.

★★☆☆☆ Find a pattern of size 2x3 that is contained.

★★☆☆☆ Find a pattern of size 2x4 that is contained.

★★★☆☆ Find a pattern of size 3x3 that is avoided.

★★★☆☆ Find a pattern of size 2x3 that is avoided.

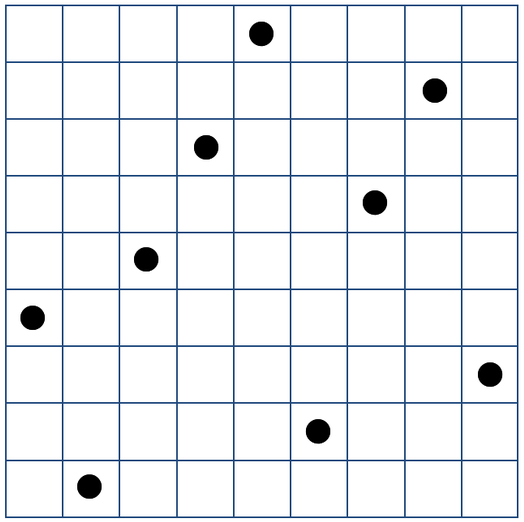
selected SOLUTIONS to part 1: Stadtpark - Bisamberg

(pages 12-28)

In case you have any questions regarding the other riddles, please contact me directly.

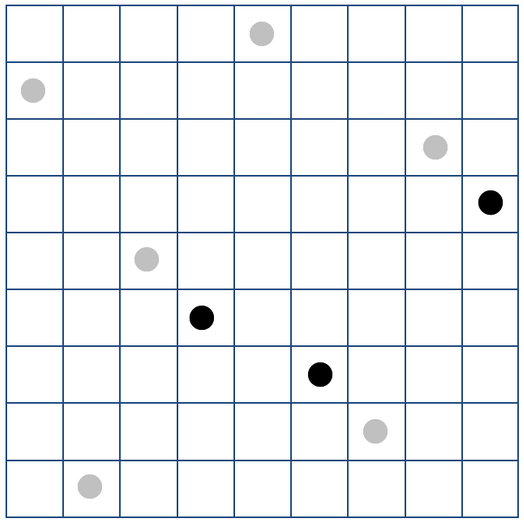
STADTPARK

There are many possibilities , for instance:



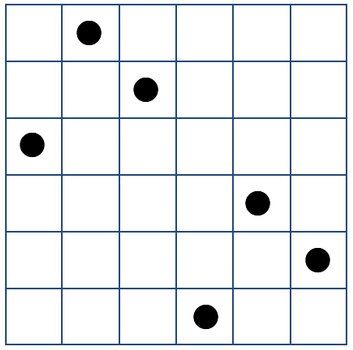
VOLKSGARTEN

Again, there are many possibilties, for instance:



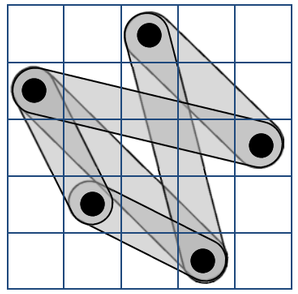
There are three possibilities for the pawn placed in the fourth row, then two for the pawn placed in sixth row and one for the pawn placed in the last one.
Makes six possibilities in total.

SCHÖNBRUNN



There are six possibilities for the pawn placed in the first row, then five for the pawn placed in the second row etc. until one possibility remains for the last row. This makes in total: 6*5*4*3*2*1=720. This can also be written as 6!.
For a grid with 15 rows and 15 columns, there are 15! =15*14*…*3*2*1= 1 307 674 368 000 possibilities!

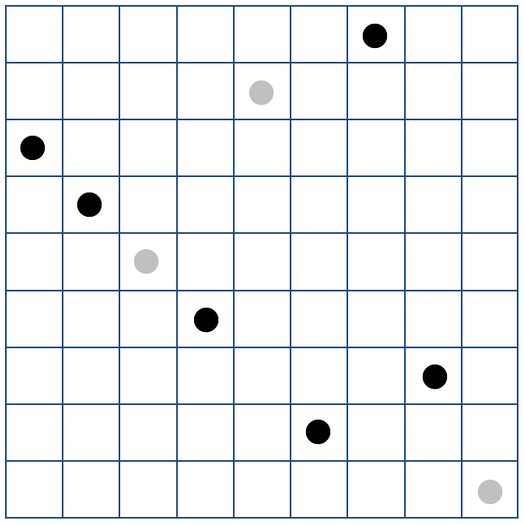
SOPHIENALPE



As seen above, the forbidden pattern appears six times in this configuration.

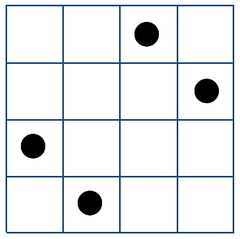
RATHAUSPARK

Again, there are many possibilities, for instance:

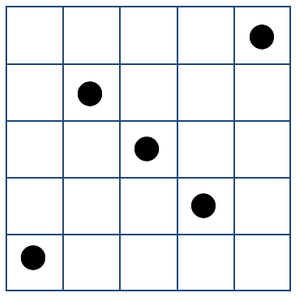


There are 6*5*4*3*2*1=720 possibilities in total.

PRATER

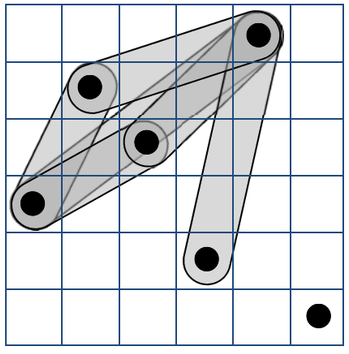


There are four possibilities for the first row, three for the second, two for the third and one for the last one.
Makes 4*3*2*1=24 possibilities in total.



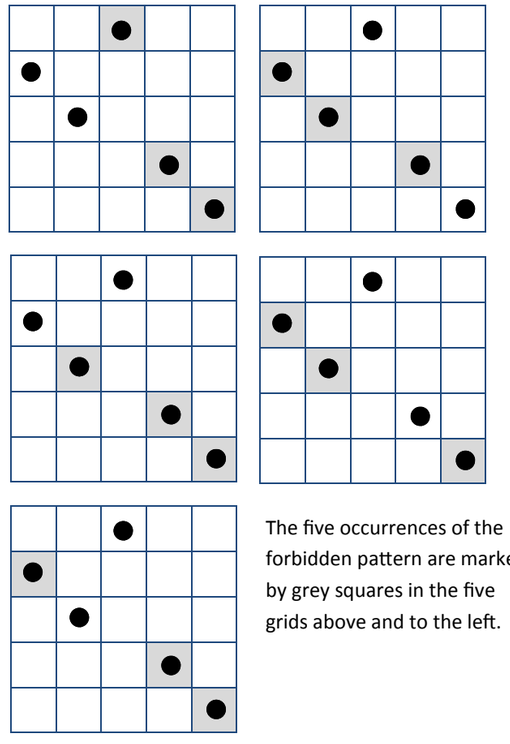
There are five possibilities for the first row and 4*3*2*1=24 possibilities for the remaining ones.
Makes 5*4*3*2*1=120 in total.

HAMEAU



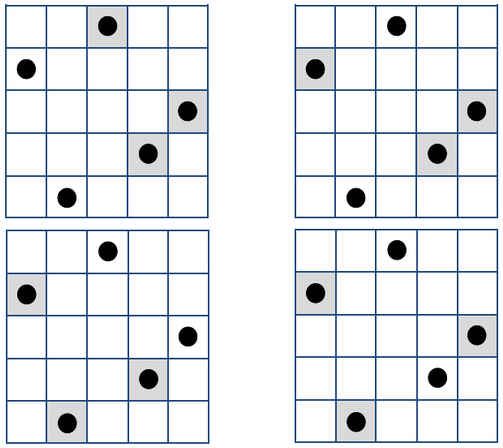
As seen above, the forbidden pattern appears six times in this configuration.

DONAUINSEL



The five occurrences of the forbidden pattern are marked by grey squares in the five grids above and to the left.

LOBAU

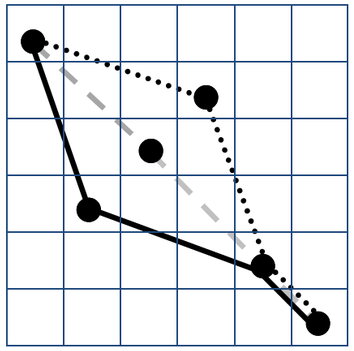


The four occurrences of the forbidden pattern are marked by grey squares in the grids above.

MARSWIESE

The forbidden pattern does not appear in the given grid.

AM HIMMEL

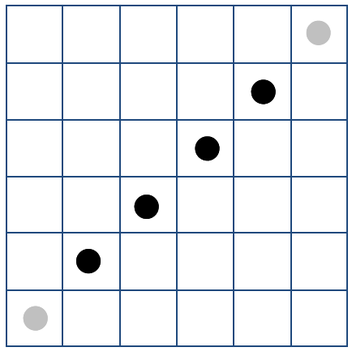


The three occurrences of the forbidden pattern by the three lines in the grid above.

COBENZL

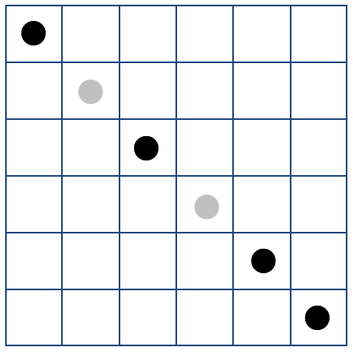
The forbidden pattern appears six times in the given grid.

KAHLENBERG



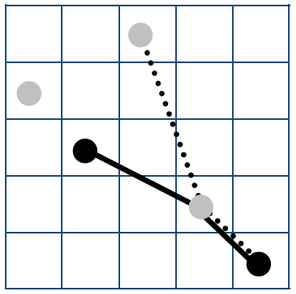
Placing the pawns as in the grid above is the only possibility of avoiding the forbidden pattern. All other configurations produce the forbidden pattern.

LEOPOLDSBERG



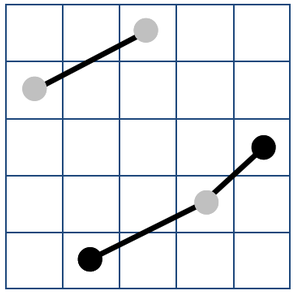
Again, there is only one possibility of avoiding the forbidden pattern. The pawns have to be placed as in the grid above.
Rotate the forbidden Kahlenberg-pattern by 90° and you will obtain the forbidden Leopoldsberg-pattern. The same holds for the grids avoiding these patterns. You may also use a horizontal or vertical reflection instead of a rotation.

STEPHANIEWARTE 1



Placing the two remaining pawns in the way above leads to several occurrences of the forbidden pattern (and this is the only possibility). Two occurrences of the forbidden pattern have been marked with lines.

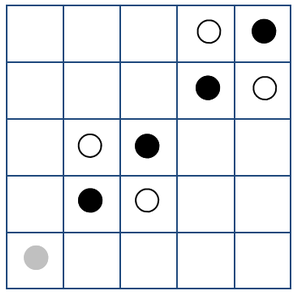
STEPHANIEWARTE 2



Placing the two remaining pawns like this leads to a configuration that avoids the forbidden pattern.

In general, if this pattern should be avoided, the pawns have to be placed so that they form (one or) two increasing sequences (=lines). In the example above these two increasing lines have been drawn in black.

HIRSCHENKOGEL

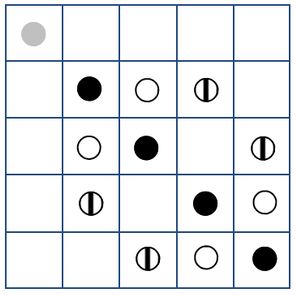


Both the configuration with black pawns and the one with white pawns (together with the initially placed grey pawn) avoid the forbidden pattern, since they consist of one or two increasing lines. See the solution Stephaniewarte 2 above. The total number of allowed configurations can be calculated with the help of *Catalan numbers*: the number of possibilities of placing the four remaining pawns is equal to the fourth Catalan number.

$$c_4 = \frac{1}{4+1}\binom{2\cdot 4}{4} = \frac{8\cdot 7\cdot 6\cdot 5}{5\cdot 4\cdot 3\cdot 2} = 14$$

Thus there are fourteen such allowed possibilities.

BISAMBERG



In the grid above, the white, striped and black pawns together with the initially placed grey pawn correspond to configurations where the forbidden pattern is avoided. Observe that the forbidden Bisamberg-pattern can be obtained by rotating the forbidden Hirschenkogel-pattern by 90° (or by reflecting it on a horizontal axis). Thus, a configuration avoiding Bisamberg can be obtained by rotating a configuration avoiding Hirschenkogel. Again, there 14 such possibilities in total.

selected

SOLUTIONS to part 2:

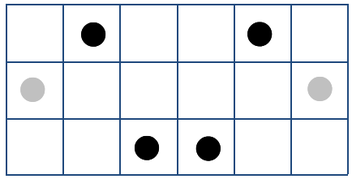
Burggarten - Hochkönig

(pages 44-53)

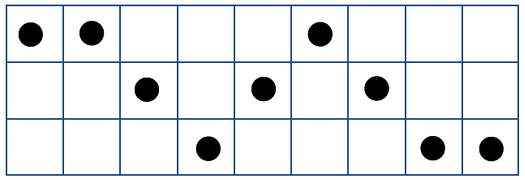
In case you have any questions regarding the other riddles,
please contact me directly.

BURGGARTEN

There are many possibilities , for instance:

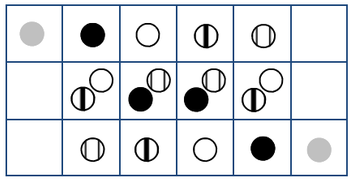


Again, there are many possibilities . Here is one:

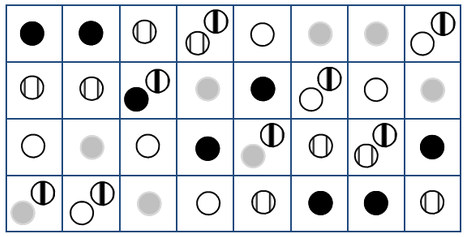


LAAER BERG

Here are four different ways of placing the remaining pawns in an allowed way, marked by different types of pawns:



Here are five ways of placing the pawns in allowed way:

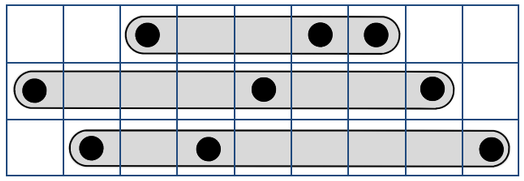


In total there are 2520 possibilities. This number can be calculated using the formula (m=2, n=4):

$$\binom{n \cdot m}{m, m, \ldots, m} = \binom{8}{2, 2, 2, 2} = \frac{8!}{2!2!2!2!} = 2520$$

GAHNS

The three occurrences of the forbidden pattern on the left have been marked in grey:

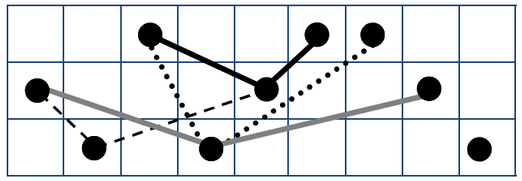


It does not matter in which (allowed) way the pawns are placed: this pattern will always appear, since there must always be three pawns in every row of the grid.

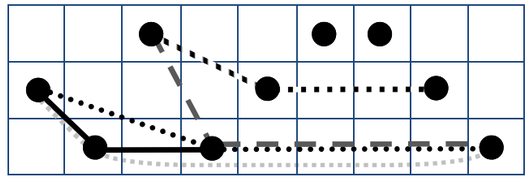
The pattern on the right however does not appear and may not appear in the grid if the pawns are placed correctly: one is not allowed to place four or more pawns in the same row.

SCHÖPFL

Four occurrences (in total there are eight) of the pattern on the left are marked with the help of lines in the grid below:

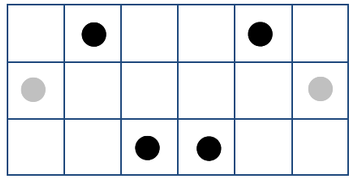


And here the five occurrences of the pattern on the right:

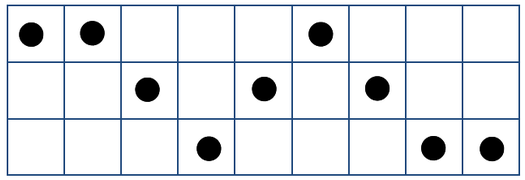


BURGGARTEN

There are many possibilities , for instance:

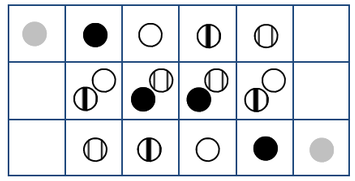


Again, there are many possibilities . Here is one:

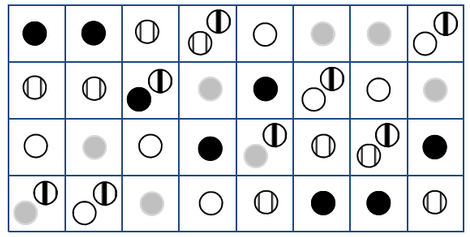


LAAER BERG

Here are four different ways of placing the remaining pawns in an allowed way, marked by different types of pawns:



Here are five ways of placing the pawns in allowed way:

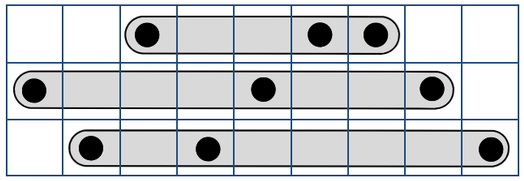


In total there are 2520 possibilities. This number can be calculated using the formula (m=2, n=4):

$$\binom{n \cdot m}{m, m, \ldots, m} = \binom{8}{2, 2, 2, 2} = \frac{8!}{2!2!2!2!} = 2520$$

GAHNS

The three occurrences of the forbidden pattern on the left have been marked in grey:

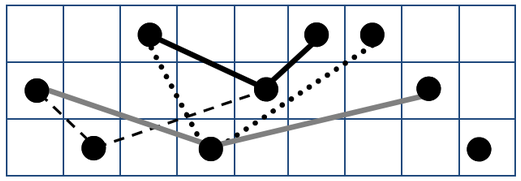


It does not matter in which (allowed) way the pawns are placed: this pattern will always appear, since there must always be three pawns in every row of the grid.

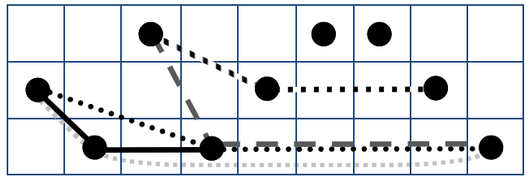
The pattern on the right however does not appear and may not appear in the grid if the pawns are placed correctly: one is not allowed to place four or more pawns in the same row.

SCHÖPFL

Four occurrences (in total there are eight) of the pattern on the left are marked with the help of lines in the grid below:

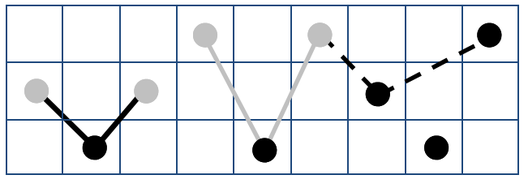


And here the five occurrences of the pattern on the right:

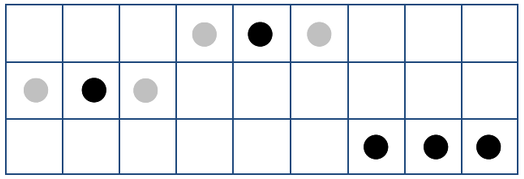


GESCHRIEBENSTEIN

If the pawns are placed in the following way, the forbidden pattern appears several times. Three occurrences have been marked:

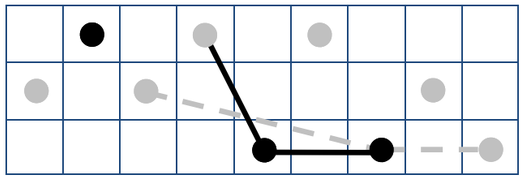


If the remaining pawns are placed in the following way, the forbidden pattern is avoided.



LINDKOGEL

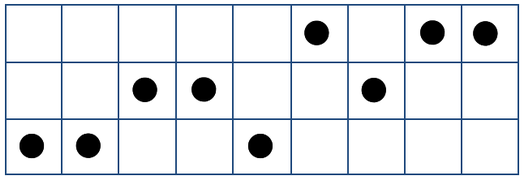
If the pawns are placed in the following way, the forbidden pattern appears more than once. Two occurrences have been marked:



There is no way of placing the remaining three pawns in a way that does not produce the forbidden pattern. You can try it out: every one of the three possible configurations leads to the forbidden pattern.
In order to avoid the forbidden pattern, two of the pawns in the last row have to be placed on the left side of the grid, i.e. in the first and second column.

HOCHKAR

One possible way of placing the eight pawns so that the forbidden pattern is avoided is given by:



The number of possibilities of placing the pawns so that the forbidden pattern is avoided can be calculated using the following formula:

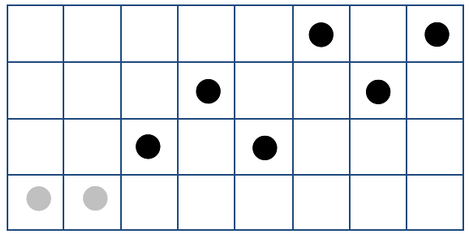
$$\prod_{k=1}^{n-1} (k \cdot m + 1)$$

Here m is the number of pawns per row and n is the number of rows. The total number of possibilities is thus:

$$(3+1)\,(3 \cdot 2 + 1) = 28$$

HOHE VEITSCH 1

Here is one possibility of placing the pawns in an allowed way:



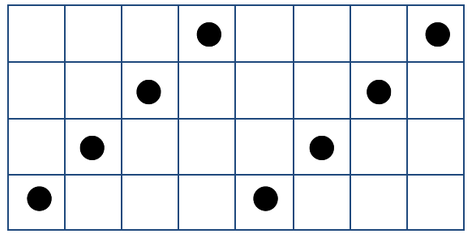
The total number of possibilities of placing the pawns so that both patterns are avoided can again be calculated with the help of the Catalan numbers. The number of remaining rows that have to be filled up is equal to three, thus the total number of possibilities is given by the third Catalan number:

$$c_3 = \frac{1}{3+1}\binom{2 \cdot 3}{3} = \frac{6 \cdot 5 \cdot 4}{4 \cdot 3 \cdot 2} = 5$$

The five possibilities are: 11223344, 11232344, 11223434, 11232434 (the one represented) and 11234234.

HOHE VEITSCH 2

If we are also allowed to move the two initial pawns there are many more possibilities of avoiding the two forbidden patterns. Here is one of them:



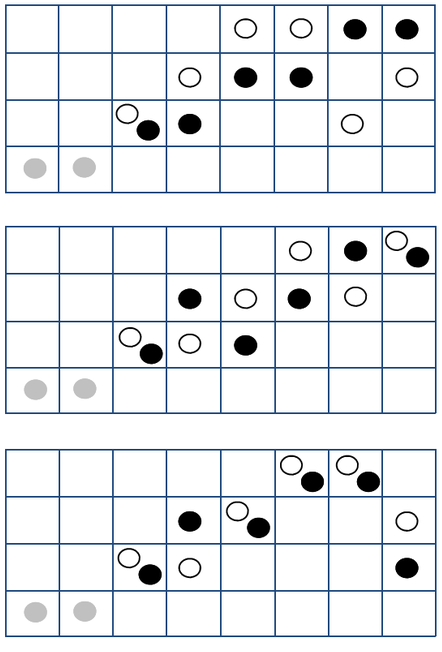
The total number of possibilities is given by the Catalan numbers. Since the total number of rows is equal to four, the total number of possibilities is given by the fourth Catalan number:

$$c_4 = \frac{1}{4+1}\binom{2 \cdot 4}{4} = \frac{8 \cdot 7 \cdot 6 \cdot 5}{5 \cdot 4 \cdot 3 \cdot 2} = 14$$

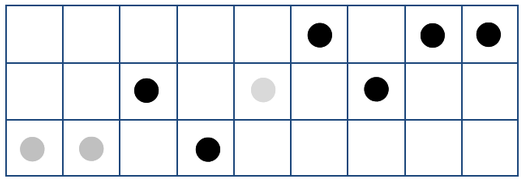
Among these 14 possibilities, 5 permutations start with 11 (the first two pawns are placed in the bottom row as in Hohe Veitsch 1) and 9 permutations start with 12 as in the grid above.

HOCHKÖNIG 1

Here are six possibilities of placing the remaining pawns in such a way that both patterns are avoided (marked by white and black pawns):

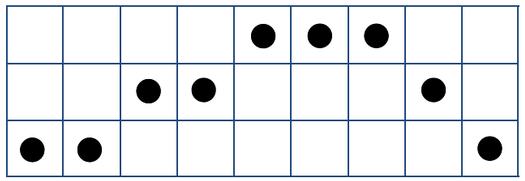


DACHSTEIN



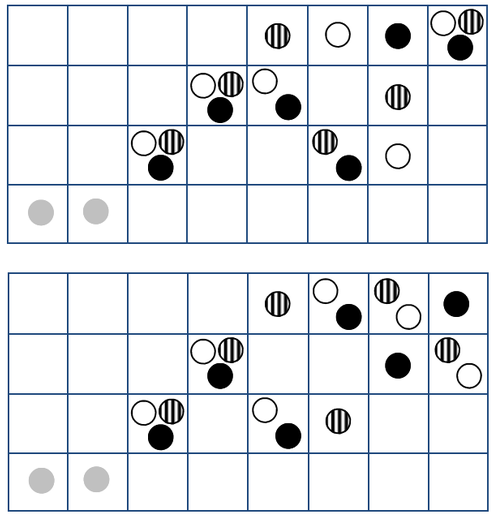
Above, one possible way of placing the remaining six pawns in order to avoid both forbidden patterns. In total, there are 22 possibilities of placing nine pawns so that both patterns are avoided. With the three pawns that were already placed, twelve possibilities remain. These are: 111222333, 112122333, 112221333, 112223133, 112223313, 112223331, 111223233, 112123233, 111223323, 112123323, 111223332 and 112123332.

Note that the first two pawns always need to be placed in the bottom row, otherwise the forbidden pattern on the left is contained. Here is one possibility of placing the pawns if the three initial pawns may be moved somewhere else:



HOCHKÖNIG 2

Being allowed to move the pawn in the first column doesn't create more possibilities (if it is placed somewhere else, the forbidden pattern on the left is automatically contained). In total, there are 12 possibilities of placing the pawns, the remaining 6 of which are represented below with black, white and striped pawns:



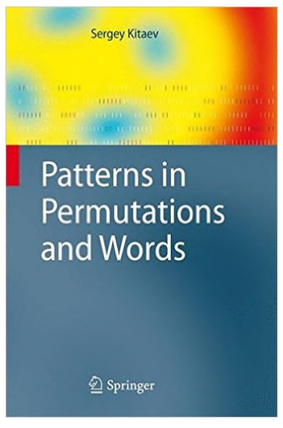
REFERENCES

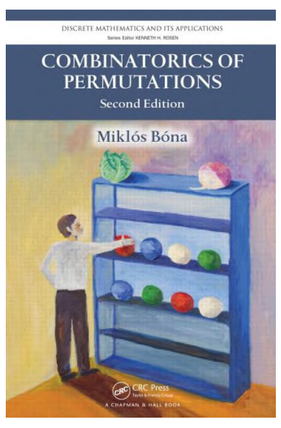
Bibliography p. 111

For the case that you want to read up on the theoretical background of MOUNTAINOUS PATTERNS and wish to find out about other aspects of permutation patterns, I have gathered some useful references here.

Textbooks/Handbooks

Sergey Kitaev's *Patterns in Permutations and Words* (Springer, 2011, ISBN 978-3-642-17332-5) is a comprehensive survey of the main results within this topic. Amongst others, it discusses generalizations of permutation patterns, describes several motivation points for the study of patterns in permutations and words and presents applications to other fields.





Miklós Bóna's *Combinatorics of permutations* (Chapman and Hall/CRC , second edition 2012, ISBN 9781439850510) is an introduction to various combinatorial aspects of permutations and is written in a very entertaining and engaging style. Chapters 4 and 5 are devoted to pattern avoidance in permutations and present many beautiful results. This book triggered my interest for permutation patterns.

The recently published *Handbook of Enumerative Combinatorics* (edited by Miklós Bóna, Chapman and Hall/CRC, 2015, ISBN 9781482220858) surveys state-of-the-art methods of combinatorial enumeration. The chapter *Permutation Classes* contributed by Vincent Vatter is devoted to current developments in the field and takes a structural view at permutation patterns. It is available online at: arxiv.org/abs/1409.5159

My own research

My webpage at marielouise.lackner.xyz contains more information about the research that I have performed so far on permutation patterns and other topics within combinatorics. Amongst others, you can find the following there:

- My master's thesis *Restricted Permutations on Multisets*, written under the supervision of Alois Panholzer, TU Wien, 2011. www.ub.tuwien.ac.at/dipl/2011/AC07810416.pdf
- A journal publication based on the results of my master's thesis: *On restricted permutations on regular multisets*, in Permutation Patterns 2012 Proceedings, Special Issue of Pure Mathematics and Applications, 24 (2): 59-82, 2013. puma.dimai.unifi.it/24_2/bruner.pdf
- My PhD thesis *Patterns in labelled combinatorial objects*, written under the supervision of Alois Panholzer, TU Wien, 2015. www.ub.tuwien.ac.at/diss/AC12313331.pdf

Other resources

- Since 2003, there is a yearly conference called *Permutation Patterns* devoted to this topic.
- Michael Albert, Univ. of Otago, created a Java-tool named *PermLab* that is intended as a laboratory for conducting research on permutation patterns (available via his webpage).

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MOUNTAINOUS PATTERNS is a collection of mathematical riddles, created by Marie-Louise Lackner to give newcomers a playful insight into the fascinating topic of pattern avoidance in permutations. Marie-Louise has been performing research on pattern restricted permutations since 2011 and is eager to make this bustling field of discrete mathematics accessible to a larger public that does not necessarily have a mathematical background.

Give it a try and have some fun with permutations!